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CONVOLUTION OF χ-ORBITAL MEASURES ON COMPLEX GRASSMANNIANS

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Abstract. Let $SU(p+q)/S\left(U(p)\times U(q)\right)$ be the Grassmannian of complex p-dimensional subspaces of $\mathbb{C}^{p+q}$, where p and q are integers such that $p\geq q\geq 2$. The aim of this paper is to extend the main result in [**3**], [**2**] to the case of convolution of χ-orbital measures where χ is a character of $S\left(U(p)\times U(q)\right)$. More precisely, we give sufficient conditions for the $\mathcal{C}^{\nu}$-smoothness of the Radon Nikodym derivative

$$f_{a_1,\ldots,a_r,\chi}=d\left(\mu_{a_1,\chi}*\ldots*\mu_{a_r,\chi}\right)/d\mu_{SU(p+q)}$$

of the convolution $\mu_{a_1,\chi}*\ldots*\mu_{a_r,\chi}$ with respect to the Haar measure $\mu_{SU(p+q)}$ of $SU(p+q)$, where $\mu_{a_j,\chi}$ are some orbital measures defined below.

1. Introduction

Let $M=U/K$ be a compact symmetric space and let $a_1,a_2,\cdots,a_r$ be elements in $U\backslash N_U(K)$, where $N_U(K)$ is the normalizer of K in U, and consider the following linear functionals defined by

$$\mathcal{I}_{a_j}\left(f\right)=\iint_{K\times K}f\left(k_1a_jk_2\right)d\mu_K\left(k_1\right)d\mu_K\left(k_2\right),\qquad f\in\mathcal{C}(U),\ j=1,\ldots,r,$$

where $\mathcal{C}\left(U\right)$ is the space of continuous functions on U, and μ_K is the Haar measure of the Lie group K normalized by $\mu_K\left(K\right)=1$. By Riesz representation theorem, to each of the functionals $\mathcal{I}_{a_j}$, $j=1,\ldots,r$ corresponds a singular measure μ_{a_j} in U supported on Ka_jK. Several authors considered the problem of the regularity of the convolution of the orbital measures μ_{a_j}. In [**10**], Ragozin proved that the convolution of r orbital measures is absolutely continuous with respect to the normalized Haar measure of U if $r\geq\dim M$. The absolute continuity of the convolutions of the measures μ_{a_j} with respect to the Haar measure μ_U of U, has also been discussed in [**4**]. The L^2 regularity of the Radon-Nikodym derivative of the convolution of the measure μ_{a_j}, where $M=SU(2)/SO(2)$, with respect to the Haar measure of $SU(2)$, has been studied in [**5**]. The smoothness of the Radon-Nikodym derivative of the convolution of the measures μ_{a_j} on some symmetric compact spaces U/K with respect to the normalized Haar measure of U, was considered in [**2**] and [**3**]. The convolution of orbital measures on arbitrary non-compact symmetric spaces G/K was studied in [**6**]. Specifically, the paper examines the L^2-regularity (resp.

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C^{ν}-smoothness) of the Radon-Nikodym derivative of the convolution $\mu_{a_1} * \cdots * \mu_{a_r}$ with respect to a fixed left Haar measure μ_G on G.

It is known that if a result is true for Grassmannians, it is likely to be true for all compact symmetric spaces. Following this guideline, we considered in [**2**] the question of smoothness of the Radon-Nikodym derivative of the convolution of the orbital measures μ_{a_j} on complex Grassmannians. In this paper, we consider the case of the complex Grassmannian $M = U/K$, where $U = SU\left(p+q\right)$ and $K = S\left(U(p) \times U(q)\right)$, with p and q being integers such that $p \geq q \geq 2$, and we introduce the following twisted complex valued functionals:

$$\mathcal{I}_{a_j,\chi}\left(f\right) = \int_K \int_K f\left(k_1 a_j k_2\right) \chi(k_1 k_2)^{-1} d\mu_K(k_1) d\mu_K(k_2), \;\; f \in \mathcal{C}\left(U\right), \, j = 1, ..., r,$$

where $\mathcal{C}\left(U\right)$ is the space of continuous functions on U, and

$$\chi : K \longrightarrow \mathbb{C}$$

is a character. Again, by Riesz representation theorem, to the functional $\mathcal{I}_{a_j,\chi}$ corresponds a $\mathbb{C}$-valued regular Borel measure $\mu_{a_j,\chi}$, $j = 1, 2, ..., r$.

Since $\operatorname{supp}(\mu_{a_j,\chi}) = Ka_jK$ for $j = 1, \ldots, r$ and Ka_jK has empty interior in U, the measures $\mu_{a_j,\chi}$ are singular.

In the first part of this paper, we give a sufficient condition for $\mu_{a_1,\chi} * \cdots * \mu_{a_r,\chi}$ to be absolutely continuous with respect to the normalized Haar measure μ_U of U, obtained in section 3. In the second part, for each given positive integer ν, a lower bound $C\left(p,q,\nu\right)$, as defined below, is provided such that for every $r \geq C(p,q,\nu)$, the Radon-Nikodym derivative

$$\frac{d\big(\mu_{a_1,\chi} * \cdots * \mu_{a_r,\chi}\big)}{d\mu_U}$$

is in $\mathcal{C}^{\nu}(U)$, the space of ν-times continuously differentiable functions on U, where μ_U is the Haar measure on U, normalized by $\mu_U(U) = 1$. More precisely, let ν be a positive integer and

$$C\left(p,q,\nu\right) := \max\left(\left\lfloor \frac{(p+q)^2 + 2\nu + q(2p-1)}{2p-q} \right\rfloor + 1, 2pq\right).$$

The aim of this paper is to prove the following

Theorem (Main Theorem)**.** Let ν be a positive integer. If

$$r \geq C\left(p,q,\nu\right),$$

then

$$\frac{d\big(\mu_{a_1,\chi} * \cdots * \mu_{a_r,\chi}\big)}{d\mu_U} \in \mathcal{C}^{\nu}\left(SU(p+q)\right).$$

The paper is organized as follows. In section 2, we collect some results about χ-spherical representations and χ-spherical functions which will be needed in the paper. In section 3, we give a sufficient condition for the absolute continuity of the convolution of the χ-orbital measures with respect to the normalized Haar measure on U. In section 4, we study the Fourier transform of the Radon-Nikodym derivative of the convolution of χ-orbital measures with respect to the Haar measure on U. In section 5, we review some of the main properties of restricted roots on complex Grassmannians. In section 6, we give a proof of the main result of the paper.

2. The χ_l-Spherical Representations

Let G be a non-compact, connected, simple Lie group, and $\mathfrak{g}$ its Lie algebra. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

be a Cartan decomposition of the Lie algebra $\mathfrak{g}$. It is known that

$$\mathfrak{u} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$$

is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m} = \mathfrak{h} \cap \mathfrak{k}$. Let U be the simply connected, simple Lie group with Lie algebra $\mathfrak{u}$ and let K be a Lie group with Lie algebra $\mathfrak{k}$.

Since every character χ of K is trivial on the commutator subgroup $[K, K]$, and since $K = Z(K)\,[K, K]$, $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$, $Z(K)$ is the center of K. We see that if $\mathfrak{z}(\mathfrak{k})$ is trivial, then every character is trivial. Therefore, we will assume in what follows that $\mathfrak{z}(\mathfrak{k})$ is not trivial, hence K is not semisimple and therefore U/K is Hermitian.

Let Z be a non-zero element of $\mathfrak{z}(\mathfrak{k})$ such that $\exp(tZ) \in Z(K)$ and $\exp(tZ) \in [K, K]$ if and only if $t \in 2\pi\mathbb{Z}$. For each integer l, put

$$\chi_l(x) = \begin{cases} 1 & \text{if } x \in [K,\, K]; \\ \\ \exp(itl) & \text{if } x = \exp(tZ). \end{cases}$$

It can be seen that for each integer l, χ_l is a character of K, moreover every character of K is of that form, i.e., the family $(\chi_l)_{l\in\mathbb{Z}}$ exhausts all the characters of K, more precisely, we have the following

Proposition 2.1 ([**11**]). *For every integer l, χ_l is a character of K. Moreover, every character of K is of the form χ_l for some integer l.*

Let $\pi_\lambda : U \longrightarrow GL(E_\lambda)$ be an irreducible unitary representation of U with highest weight λ. For a non-trivial character χ_l of K, let

$$E_\lambda^l = \{X \in E_\lambda \,|\, \pi_\lambda(k)(X) = \chi_l(k)\, X \text{ for all } k \in K\}. \tag{2.1}$$

The representation (π_λ, E_λ) of U is said to be χ_l-spherical if $E_\lambda^l \neq 0$. The spherical representations correspond to the case $l = 0$.

Fix an integer l in $\mathbb{Z}$, from now on we will denote by $\widehat{U}_l$ the set of χ_l-spherical irreducible unitary representations of U and $\Lambda^+(U)$ the set of highest weights of irreducible representation of U. It is well known that if U/K is a symmetric space and (π_λ, E_λ) is a χ_l-spherical representation, then $\dim_{\mathbb{C}} E_\lambda^l = 1$. The set of highest weights of χ_l-spherical representations of U will be denoted by $\Lambda_l^+(U)$.

Recall that a function φ on U is called (an elementary) χ_l-spherical function if it is not identically zero, and if it satisfies the following conditions:

$$\varphi(k_1 u k_2) = \chi_l(k_1 k_2)^{-1}\varphi(u), \text{ for all } u \in U,\ k_1, k_2 \in K, \tag{2.2}$$

and

$$\int_K \varphi(u_1 k u_2)\chi_l(k)\, d\mu_K(k) = \varphi(u_1)\varphi(u_2), \text{ for all } u_1, u_2 \in U. \tag{2.3}$$

Actually it can be proved that (2.2) is a consequence of (2.3), see [**8**].

For $\lambda \in \Lambda_l^+(U)$, let $(.,.)_\lambda$ be a U-invariant inner product on E_λ, i.e.,

$$(\pi_\lambda(u)X, \pi_\lambda(u)Y)_\lambda = (X,Y)_\lambda .$$

for all $u \in U$ and $X, Y \in E_\lambda$, and let $\|\cdot\|_\lambda$ the corresponding norm. Choose an orthonormal basis $\{e_1^\lambda, \cdots, e_{d_\lambda}^\lambda\}$ of E_λ, such that $e_1^\lambda \in E_\lambda^l$, where $d_\lambda = \dim E_\lambda$.

It is well known that $\psi_{\lambda,l}(u) = \left(e_1^\lambda, \pi_\lambda(u)e_1^\lambda\right)_\lambda$ is an elementary χ_l- spherical function and conversely any elementary χ_l- spherical function on U is of this form.

3. Absolute Continuity of Convolutions of χ_l-Orbital Measures

Let $a \in U \backslash N_U(K)$, and let $\mathcal{I}_{a,\chi_l}$ be as in section 1. By the Riesz representation theorem, to the functional $\mathcal{I}_{a,\chi_l}$ corresponds a unique $\mathbb{C}$-valued regular Borel measure μ_{a,χ_l} on U, such that

$$\mathcal{I}_{a,\chi_l}(f) = \int_U f(g)d\,\mu_{a,\chi_l}(g), \quad f \in \mathcal{C}(U).$$

Fix a set of points $a_1, \cdots, a_r$ in $U \backslash N_U(K)$ and consider the linear functional

$$\mathcal{I}_{a_1,a_2,\cdots,a_r,\chi_l}(f) = \int_K \cdots \int_K f\big(k_1a_1k_2\cdots k_ra_rk_{r+1}\big)\chi_l\,(k_1\cdots k_{r+1})^{-1}\,d\mu_K(k_1)\cdots d\mu_K(k_{r+1}).$$

Again, by the Riesz representation theorem, to the functional $\mathcal{I}_{a_1,a_2,\cdots,a_r,\chi_l}$ corresponds a measure $\mu_{a_1,\cdots,a_r,\chi_l}$, such that

$$\mathcal{I}_{a_1,a_2,\cdots,a_r,\chi_l}(f) = \int_U f(g)d\,\mu_{a_1,\cdots,a_r,\chi_l}(g)$$

for all f in $\mathcal{C}(U)$.

Proposition 3.1.

$$\mu_{a_1,a_2,\cdots,a_r,\chi_l} = \mu_{a_1,\chi_l} * \cdots * \mu_{a_r,\chi_l}.$$

Proof. Let $f \in \mathcal{C}(U)$ and let $r = 2$. Since μ_K is normalized, we have

$$\begin{aligned}
\int_U f(g)\,d\mu_{a_1,\chi_l} * \mu_{a_2,\chi_l}(g) &= \int_U\int_U f(g_1g_2)d\mu_{a_1,\chi_l}(g_1)d\mu_{a_2,\chi_l}(g_2)\\
&= \int_K\int_K\int_K\int_K f(k_1a_1k_2k_3a_2k_4)\chi_l^{-1}(k_1k_2k_3k_4)\,d\mu_K(k_1)d\mu_K(k_2)d\mu_K(k_3)d\mu_K(k_4)\\
&= \int_K\int_K\int_K\left(\int_K f(k_1a_1k_2k_3a_2k_4)\chi_l^{-1}(k_1k_2k_3k_4)\,d\mu_K(k_2)\right)d\mu_K(k_1)d\mu_K(k_3)d\mu_K(k_4)\\
&= \int_K\int_K\int_K\left(\int_K f(k_1a_1k_5a_2k_4)\chi_l^{-1}(k_1k_5k_4)\,d\mu_K(k_5)\right)d\mu_K(k_1)d\mu_K(k_3)d\mu_K(k_4)\\
&= \int_K\int_K\int_K f(k_1a_1k_2a_2k_3)\chi_l^{-1}(k_1k_2k_3)\,d\mu_K(k_1)d\mu_K(k_2)d\mu_K(k_3)\\
&= \int_U f(g)\,d\mu_{a_1,a_2,\chi_l}(g).
\end{aligned}$$

By induction, we get

$$\int_U f(g)d\big(\mu_{a_1,\chi_l} * \cdots * \mu_{a_r,\chi_l}\big)(g) = \int_U f(g)\, d\mu_{a_1,a_2,\cdots,a_r,\chi_l},$$

for all $f \in \mathcal{C}(U)$. Hence the proposition. $\square$

Lemma 3.2. *For every Borel subset $\mathscr{B}$ of U, we have*

(i) *that*

$$\mu_{a,\chi_l}(\mathscr{B}) = \int_U \mathbb{1}_{\mathscr{B}}(x)\, d\mu_{a,\chi_l}(x) = \int_K \int_K \mathbb{1}_{\mathscr{B}}(k_1 a k_2)\, \chi_l (k_1 k_2)^{-1}\, d\mu_K(k_1)\, d\mu_K(k_2),$$

(ii) *and*

$$\int_U \mathbb{1}_{\mathscr{B}}(x)\, d\mu_{a_1,\dots,a_r,\chi_l} =$$
$$\int_K \cdots \int_K \mathbb{1}_{\mathscr{B}}(k_1 a_1 k_2 \cdots k_r a_r k_{r+1})\, \chi_l(k_1 \cdots k_{r+1})^{-1} d\mu_K(k_1) \cdots d\mu_K(k_{r+1}),$$

where $\mathbb{1}_{\mathscr{B}}$ is the characteristic function of $\mathscr{B}$.

Proof. Consider the continuous mapping $\tau_a : K \times K \longrightarrow U$ such that

$$\tau_a(k_1, k_2) = k_1 a k_2.$$

Let λ_a be the push-forward of the product measure $\mu_K \times \mu_K$ on $K \times K$ under the map τ_a, i.e., for a Borel subset $\mathscr{B}$ of U,

$$\lambda_a(\mathscr{B}) = (\tau_a)_* (\mu_K \times \mu_K)(\mathscr{B}) = \big(\mu_K \times \mu_K\big)\big(\tau_a^{-1}(\mathscr{B})\big).$$

Hence

$$\lambda_a(\mathscr{B}) = \int_K \int_K \mathbb{1}_{\mathscr{B}}(k_1 a k_2)\, d\mu_K(k_1)\, d\mu_K(k_2).$$

By (Corollary 1 [**12**]), λ_a is regular. Define the complex measure λ_{a,χ_l} on U by

$$\lambda_{a,\chi_l}(\mathscr{B}) = \int_K \int_K \mathbb{1}_{\mathscr{B}}(k_1 a k_2)\, \chi_l(k_1 k_2)^{-1}\, d\mu_K(k_1)\, d\mu_K(k_2).$$

Fix a Borel set $\mathscr{B}$, and let $\epsilon > 0$. Since λ_a is regular, there exist an open set O and a compact set C of U such that $C \subseteq \mathscr{B} \subseteq O$ and

$$\lambda_a(\mathscr{B} \backslash C) < \epsilon \quad \text{and} \quad \lambda_a(O \backslash \mathscr{B}) < \epsilon.$$

Therefore

$$\begin{aligned} |\lambda_{a,\chi_l}|(\mathscr{B} \backslash C) &= \sup \sum_{i=1}^{n} |\lambda_{a,\chi_l}(\mathscr{B}_i)| \\ &\leq \sup \sum_{i=1}^{n} \lambda_a(\mathscr{B}_i) \\ &= \lambda_a(\mathscr{B} \backslash C) \\ &< \epsilon, \end{aligned}$$

where the supremum is taken over all partitions of $\mathscr{B} \backslash C$ into a finite number of Borel sets $\mathscr{B}_1, \dots \mathscr{B}_n$. Similarly $|\lambda_{a,\chi_l}|(O \backslash \mathscr{B}) < \epsilon$. Hence λ_{a,χ_l} is regular. A similar argument as in (Theorem 2, [**12**]), shows that for any Borel set $\mathscr{B}$,

$\lambda_{a,\chi_l}(\mathscr{B}) = \mu_{a,\chi_l}(\mathscr{B})$, which proves part (i) of the Lemma. Part (ii) of the Lemma follows from ([**12**], Theorem 2). $\square$

Proposition 3.3. *Let $a_i \in U \backslash N_U(K)$, $i = 1, \ldots, r$ and $\psi^r : K^{r+1} \longrightarrow U$, be such that*

$$\psi^r(k_1, \ldots, k_{r+1}) = k_1 a_1 k_2 \ldots k_r a_r k_{r+1}.$$

Then Rank $\psi^r \geq \min(\dim U, r + \dim K)$, except on a proper analytic subvariety of K^{r+1}.

Proof. By [**13**], the pair (U, K) satisfies condition (2.1) in [**10**]. So the Proposition follows from ([**10**], Proposition 2.2). $\square$

Theorem 3.4. *Let (U, K) and μ_{a_i, χ_l}, $i = 1, \ldots, r$ be as above. If $r \geq \dim U/K$ then $\mu_{a_1,\ldots,a_r,\chi_l}$ is absolutely continuous with respect to the Haar measure μ_U of U.*

Proof. Let $S \subseteq U$ be a measurable set and $\mathbb{1}_S$ its characteristic function. Assume that $\mu_U(S) = 0$, it will be shown that $\mu_{a_1,\ldots,a_r,\chi_l}(S) = 0$. By Lemma 3.2 we have

$$\begin{aligned}
|\mu_{a_1,\ldots,a_r,\chi_l}(S)| &= \left| \int_K \cdots \int_K \mathbb{1}_S(k_1 a_1 k_2 \ldots a_r k_{r+1}) \chi_l(k_1 \ldots k_{r+1})^{-1} \, d\mu_K(k_1) \ldots d\mu_K(k_{r+1}) \right| \\
&\leq \int_K \cdots \int_K \mathbb{1}_S(k_1 a_1 k_2 \ldots a_r k_{r+1}) \, d\mu_K(k_1) \ldots d\mu_K(k_{r+1}) \\
&\leq \mu_{K^{r+1}}((\psi^r)^{-1}(S)),
\end{aligned}$$

where $\mu_{K^{r+1}}$ is the Haar measure of K^{r+1}. To finish the proof we need to show that

$$\psi^r_* \left(\mu_{K_{r+1}}\right)(S) = \mu_{K^{r+1}}\left((\psi^r)^{-1}(S)\right) = 0.$$

By Proposition 3.3, ψ^r has full rank, i.e., Rank $\psi^r = \dim U = \dim K + \dim U/K$, except on a proper analytic subvariety of K^{r+1}. This subvariety has measure zero, since it has smaller dimension than K^{r+1}.
Assume that K^{r+1} (resp. U) has dimension s (resp. m). Note that $s \geq m$ and $(\psi^r)^{-1}(S)$ is closed.

Consider $p' \in (\psi^r)^{-1}(S)$ and let $p = \psi^r(p')$. By the submersion theorem there are charts (τ_1, H') centered at p' and a chart (τ_2, H) centered at p such that $\psi^r(H') \subseteq H$,

$$\tau_2 \circ \psi^r \circ \tau_1^{-1}(x, y) = x, \quad \text{for all } x \in \mathbb{R}^m,\ y \in \mathbb{R}^{s-m},\ (x, y) \in \tau_1(H).$$

Let λ_m be the Lebesgue measure of $\mathbb{R}^m$. Since the Haar measure on any Lie group is mutually absolutely continuous with respect to Lebesgue measure in any coordinate patch we have

$$\lambda_m(\tau_2(H \cap S)) = 0.$$

Let $A \subseteq \mathbb{R}^s$ be such that

$$A = \tau_1(H' \cap (\psi^r)^{-1}(S)).$$

It follows that

$$A \subseteq \{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_s) \in \mathbb{R}^s \mid (x_1, \ldots, x_m) \in \tau_2(H \cap S)\}.$$

By Fubini's Theorem, we have

$$\lambda_s(A) = \int_{\mathbb{R}^s} \mathbb{1}_A(x) \, d\lambda_s(x)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^l} \int_{\mathbb{R}^m} \mathbb{1}_A(x,y)\, d\lambda_m(x) d\lambda_l(x) \\
&\leq \int_{\mathbb{R}^l} \int_{\mathbb{R}^m} \mathbb{1}_{\tau_2(H\cap S)}(x)\, d\lambda_m(x) d\lambda_l(x) \\
&= \int_{\mathbb{R}^l} \lambda_m(\tau_2(H\cap S))\, d\lambda_l(x) \\
&= 0.
\end{aligned}
$$

Thus for any $x \in K^{r+1}$, there exists a coordinate neighborhood $H' \subseteq K^{r+1}$ of x such that

$$\mu_{K^{r+1}}(H' \cap (\psi^r)^{-1}(S)) = 0.$$

By compactness of K^{r+1}, we can find a finite coordinate neighborhoods $\{H_i'\}_{i=1}^d$ such that

$$(\psi^r)^{-1}(S) \subseteq \bigcup_{i=1}^d H_i',$$

so we have

$$(\psi^r)^{-1}(S) = \bigcup_{i=1}^d (H_i' \cap (\psi^r)^{-1}(S)).$$

Therefore

$$\mu_{K^{r+1}}\left((\psi^r)^{-1}(S)\right) \leq \sum_{i=1}^d \mu_{K^{r+1}}(H_i' \cap (\psi^r)^{-1}(S)) = 0.$$

Hence the Theorem. □

4. Fourier Transform of the Radon-Nikodym Derivative $f_{a_1,\dots,a_r,\chi_l}$

Let $\chi_l : K \longrightarrow GL(E_{\chi_l})$ be a representation of K. Define an action of K on $U \times E_{\chi_l}$ from the right by

$$(u, X)\, k = \left(uk, \chi_l\left(k^{-1}\right) X\right), \quad u \in U,\ X \in E_{\chi_l}.$$

Define an equivalence relation $\mathcal{R}$ on $U \times E_{\chi_l}$ by

$$(u_1, X_1)\, \mathcal{R}\, (u_2, X_2) \Leftrightarrow \exists\, k \in K \text{ such that } u_2 = u_1 k \text{ and } X_2 = \chi_l\left(k^{-1}\right) X_1.$$

The equivalence class of an element (u, X) of $U \times E_{\chi_l}$ will be denoted by $[u, X]$ and the set of equivalence classes by $\mathbb{E}_{\chi_l} = U \times_K E_{\chi_l}$, i.e.,

$$\mathbb{E}_{\chi_l} = U \times_K E_{\chi_l} = \{[u, X] \mid u \in U \text{ and } X \in E_{\chi_l}\}.$$

Then $\mathbb{E}_{\chi_l}$ is a complex vector bundle over U/K of rank equal to $\dim_{\mathbb{C}} E_{\chi_l}$. In case χ_l is a character, i.e., $\dim_{\mathbb{C}} E_{\chi_l} = 1$, or equivalently $E_{\chi_l} \cong \mathbb{C}$, then $\mathbb{E}_{\chi_l}$ is a line bundle, which we will denote, following tradition, by $\mathbb{L}_{\chi_l}$. It is easy to see that the smooth sections $\Gamma(U/K, \mathbb{L}_{\chi_l})$ of $\mathbb{L}_{\chi_l}$ can be identified with smooth functions f on U satisfying

$$f(uk) = \chi_l(k)^{-1} f(u), \text{ for all } k \in K \text{ and } u \in U.$$

Let

$$L^2(U/K, \mathbb{L}_{\chi_l}) = \left\{f \in L^2(U) \mid f(uk) = \chi_l(k^{-1}) f(u), \text{ for all } k \in K\right\},$$

and let $L^2(U//K, \mathbb{L}_{\chi_l})$ be the subspace in $L^2(U/K, \mathbb{L}_{\chi_l})$ of functions satisfying

$$f(k_1 u k_2) = \chi_l(k_1 k_2)^{-1} f(u), \qquad \text{for all } k_1, k_2 \in K.$$

Let $\lambda \in \Lambda_l^+(U)$ and, as above, let $\pi_\lambda : U \longrightarrow GL(E_\lambda)$ be the irreducible unitary representation of U with highest weight λ. For $f \in L^1(U)$, consider the bilinear form

$$\Psi_\lambda : E_\lambda \times E_\lambda \longrightarrow \mathbb{C}$$

$$(X,Y) \longmapsto \Psi_\lambda(X,Y) = \int_U f(u)\,\big(\pi_\lambda(u)\,X, Y\big)_\lambda\, d\mu_U(u), \qquad X, Y \in E_\lambda.$$

Since

$$|\Psi_\lambda(X,Y)| \;=\; \left|\int_U f(u)\,\big(\pi_\lambda(u)\,X, Y\big)_\lambda\, d\mu_U(u)\right| \;\le\; \|X\|_\lambda\,\|Y\|_\lambda\,\|f\|_{L^1(U)},$$

it follows that Ψ_λ is continuous. By Riesz representation theorem, we deduce that there exists a continuous linear operator

$$\mathcal{F}\big(f\big)(\lambda)\colon E_\lambda \;\longrightarrow\; E_\lambda$$

such that

$$\big(\mathcal{F}\big(f\big)(\lambda)(X), Y\big)_\lambda \;=\; \int_U f(u)\,\big(\pi_\lambda(u)\,X, Y\big)_\lambda\, d\mu_U(u).$$

The operator $\mathcal{F}\big(f\big)(\lambda)$ is called the *Fourier transform* of f and for simplicity we write

$$\mathcal{F}\big(f\big)(\lambda)\,(X) = \int_U f(u)\,\pi_\lambda(u)\,(X)\; d\mu_U(u).$$

Recall that for a nontrivial character χ_l of K, we denoted by E_λ^l the subspace of E_λ defined in (2.1). Similarly, as above, there exists a continuous linear operator

$$P_{l,\lambda}\colon E_\lambda \;\longrightarrow\; E_\lambda$$

such that

$$\big(P_{l,\lambda}(X),\, Y\big)_\lambda \;=\; \int_K \chi_l\big(k^{-1}\big)\,\big(\pi_\lambda(k)\,X,\; Y\big)_\lambda\, d\mu_K(k), \quad X, Y \in E_\lambda.$$

For simplicity, we will write

$$P_{l,\lambda}(X) = \int_K \chi_l(k^{-1})\pi_\lambda(k)X\; d\mu_K(k), \quad X \in E_\lambda.$$

Recall that $\{e_1^\lambda, \cdots, e_{d_\lambda}^\lambda\}$ is an orthonormal basis of E_λ, where $e_1^\lambda \in E_\lambda^l$ and

$$d_\lambda = \dim E_\lambda.$$

It can be shown that

$$\pi_\lambda(k)\,P_{l,\lambda}(X) = \chi_l(k)\,P_{l,\lambda}(X),$$

for every $k \in K$ and $X \in E_\lambda$. Thus $P_{l,\lambda}(X) \in E_\lambda^l$. Moreover, the operator $P_{l,\lambda}$ is both self-adjoint and idempotent. Hence, it is the orthogonal projection of E_λ onto the subspace E_λ^l. Consequently,

$$\big(P_{l,\lambda}(e_j^\lambda),\, e_1^\lambda\big)_\lambda = 0, \qquad \text{for } j = 2, 3, \ldots, d_\lambda.$$

Lemma 4.1. *Let $f \in L^1(U/K;\chi_l)$. Then:*
(i) *For each $\lambda \in \Lambda^+(U)$, we have*

$$\mathcal{F}(f)(\lambda) = \mathcal{F}(f)(\lambda)\,P_{l,\lambda}.$$

(ii) *For* $\lambda \in \Lambda^+(U) \setminus \Lambda_l^+(U)$, *we have*

$$\mathcal{F}(f)(\lambda)X = 0, \quad \textit{for all } X \in E_\lambda.$$

(iii)

$$\mathrm{Tr}\big(\mathcal{F}(f)(\lambda)\big) = \big(\mathcal{F}(f)(\lambda)e_1^\lambda,\, e_1^\lambda\big)_\lambda.$$

Proof. (i) For $f \in L^1(U/K;\chi_l)$ and any $X \in E_\lambda$, we compute:

$$\begin{aligned}
\mathcal{F}(f)(\lambda)\,(P_{l,\lambda}(X)) &= \int_U f(u)\,\pi_\lambda(u)\left(\int_K \chi_l(k^{-1})\pi_\lambda(k)X\,d\mu_K(k)\right)d\mu_U(u)\\
&= \int_K\int_U \chi_l(k^{-1})f(u)\,\pi_\lambda(uk)X\,d\mu_U(u)\,d\mu_K(k)\\
&= \int_K\left(\int_U f(uk)\pi_\lambda(uk)X\,d\mu_U(u)\right)d\mu_K(k)\\
&= \int_K\left(\int_U f(u)\pi_\lambda(u)X\,d\mu_U(u)\right)d\mu_K(k)\\
&= \int_U f(u)\pi_\lambda(u)X\,d\mu_U(u) = \mathcal{F}(f)(\lambda)(X).
\end{aligned}$$

Hence,

$$\mathcal{F}(f)(\lambda) = \mathcal{F}(f)(\lambda)\,P_{l,\lambda}.$$

(ii) Since $\lambda \notin \Lambda_l^+(U)$, $E_\lambda^l = 0$. Thus

$$\mathcal{F}(f)(\lambda)X = 0, \text{ for all } X \in E_\lambda.$$

(iii) Since $P_{l,\lambda}$ is the orthogonal projection onto the one-dimensional space spanned by e_1^λ, for $X \in E_\lambda$,

$$P_{l,\lambda}(X) = (X, e_1^\lambda)_\lambda\, e_1^\lambda.$$

Thus,

$$\begin{aligned}
\mathcal{F}(f)(\lambda)(X) &= \mathcal{F}(f)(\lambda)(P_{l,\lambda}(X))\\
&= \mathcal{F}(f)(\lambda)\left((X, e_1^\lambda)_\lambda\, e_1^\lambda\right)\\
&= (X, e_1^\lambda)_\lambda\, \mathcal{F}(f)(\lambda)(e_1^\lambda).
\end{aligned}$$

It follows that

$$\mathrm{Tr}\,(\mathcal{F}(f)(\lambda)) = \big(\mathcal{F}(f)(\lambda)(e_1^\lambda),\, e_1^\lambda\big)_\lambda.$$

□

Denote by $(\cdot,\cdot)_{HS}$ the Hilbert–Schmidt inner product on $\mathrm{End}\,(E_\lambda)$, i.e.,

$$\begin{aligned}
(T,S)_{HS} &= \mathrm{Tr}\,(S^* \circ T)\\
&= \sum_{i=1}^{d_\lambda}\left(Te_i^\lambda, Se_i^\lambda\right)_\lambda,
\end{aligned}$$

where $T, S \in \mathrm{End}\,(E_\lambda)$, $d_\lambda = \dim E_\lambda$, $\{e_i^\lambda\}_{i=1}^{d_\lambda}$ an orthonormal basis of E_λ and S^* the adjoint of S. It can be proved that this definition is independent of the choice of the orthonormal basis of E_λ. Let $\|.\|_{HS}$ the corresponding norm.

Theorem 4.2 (Plancherel Theorem). *Let $f, f_1, f_2 \in L^2(U/K, \mathbb{L}_{\chi_l})$. Then*

$$(f_1, f_2)_{L^2(U)} = \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda \left(\mathcal{F}(f_1)(\lambda), \mathcal{F}(f_2)(\lambda)\right)_{HS}.$$

In particular,

$$\|f\|^2_{L^2(U)} = \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda \|\mathcal{F}(f)(\lambda)\|^2_{HS}.$$

Suppose that $\mu_{a_1,\dots,a_r,\chi_l}$ is absolutely continuous with respect to the Haar measure μ_U on U, and denote by $f_{a_1,\dots,a_r,\chi_l}$ the Radon-Nikodym derivative of $\mu_{a_1,\dots,a_r,\chi_l}$ with respect to μ_U, i.e.,

$$f_{a_1,\dots,a_r,\chi_l} d\mu_U = d\mu_{a_1,\dots,a_r,\chi_l}.$$

The following result will be used in the proof of Proposition 4.5.

Lemma 4.3.

$$\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda = \left(\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda, e_1^\lambda\right)_\lambda e_1^\lambda.$$

Proof. Note first that for each $k \in K$, the measures $f_{a_1,\dots,a_r,\chi_l}(ku)\, d\mu_U(u)$ and $\chi_l(k)^{-1} f_{a_1,\dots,a_r,\chi_l}(u)\, d\mu_U(u)$ are equal, i.e.,

$$f_{a_1,\dots,a_r,\chi_l}(ku)\, d\mu_U(u) = \chi_l(k)^{-1} f_{a_1,\dots,a_r,\chi_l}(u)\, d\mu_U(u) \tag{4.1}$$

for if $\phi \in \mathcal{C}(U)$, then

$$\begin{aligned}
&\int_U \phi(u) f_{a_1,\dots,a_r,\chi_l}(ku)\, d\mu_U(u) = \int_U \phi\left(k^{-1}u\right) f_{a_1,\dots,a_r,\chi_l}(u)\, d\mu_U(u)\\
&\quad = \int_K \cdots \int_K \phi\big(k^{-1}k_1a_1k_2\cdots k_ra_rk_{r+1}\big)\chi_l\left(k_1\cdots k_{r+1}\right)^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1})\\
&\quad = \int_K \cdots \int_K \phi\big(k_1a_1k_2\cdots k_ra_rk_{r+1}\big)\chi_l\left(kk_1\cdots k_{r+1}\right)^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1})\\
&\quad = \int_U \chi_l(k)^{-1}\phi(u) f_{a_1,\dots,a_r,\chi_l}(u)\, d\mu_U(u).
\end{aligned}$$

Therefore, from (4.1), we deduce that

$$\begin{aligned}
\pi_\lambda(k)\Big[\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda\Big] &= \int_U f_{a_1,\dots,a_r,\chi_l}(u)\,\pi_\lambda(ku)\,e_1^\lambda\, d\mu_U(u)\\
&= \int_U f_{a_1,\dots,a_r,\chi_l}\left(k^{-1}u\right)\pi_\lambda(u)\,e_1^\lambda\, d\mu_U(u)\\
&= \int_U \chi_l(k) f_{a_1,\dots,a_r,\chi_l}(u)\,\pi_\lambda(u)\,e_1^\lambda\, d\mu_U(u)\\
&= \chi_l(k)\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda.
\end{aligned}$$

Thus $\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda \in E_\lambda^l$. Since $\dim E_\lambda^l = 1$, we deduce that

$$\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda = c\, e_1^\lambda,$$

for some constant c, which is easily seen to be $\left(\mathcal{F}(f_{a_1,\dots,a_r,\chi_l})(\lambda)e_1^\lambda, e_1^\lambda\right)_\lambda$. Hence the Lemma. □

Proposition 4.4.
$$\mathrm{Tr}(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda)) = \prod_{i=1}^{r} \psi_{\lambda,l}\left(a_i^{-1}\right). \tag{4.2}$$

Proof. Observe that

$$\begin{aligned}
&\mathrm{Tr}(\mathcal{F}(f_{a_1,\ldots,a_r,\chi_l})(\lambda)) = \left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda)e_1^{\lambda}, e_1^{\lambda}\right)_{\lambda} \qquad \text{(By Lemma 4.1)}\\
&= \int_U f_{a_1,\ldots,a_r,\chi_l}(u)\left(\pi_\lambda(u)\, e_1^{\lambda}, e_1^{\lambda}\right)_{\lambda} d\mu_U(u)\\
&= \int_U f_{a_1,\ldots,a_r,\chi_l}(u)\, \psi_{\lambda,l}(u^{-1}) d\mu_U(u)\\
&= \int_U \overline{\psi_{\lambda,l}(u)}\, d\mu_{a_1,\ldots,a_r,\,\chi_l}(u) \qquad (4.3)\\
&= \int_K \cdots \int_K \overline{\psi_{\lambda,l}\left(k_1 a_1 k_2 \cdots k_r a_r k_{r+1}\right)}\chi_l(k_1 \ldots k_{r+1})^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1})\\
&= \int_K \cdots \int_K \psi_{\lambda,l}\left((k_1 a_1 k_2 \cdots k_r a_r k_{r+1})^{-1}\right)\chi_l(k_1 \ldots k_{r+1})^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1}) \qquad (4.4)\\
&= \int_K \cdots \int_K \psi_{\lambda,l}\left(k_{r+1}^{-1} a_r^{-1} k_r^{-1} \cdots k_2^{-1} a_1^{-1} k_1^{-1}\right)\chi_l(k_1 \ldots k_{r+1})^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1})\\
&= \int_K \cdots \int_K \psi_{\lambda,l}\left(a_r^{-1} k_r^{-1} \cdots k_2^{-1} a_1^{-1}\right)\chi_l(k_2 \ldots k_r)^{-1} d\mu_K(k_1)\cdots d\mu_K(k_{r+1}),
\end{aligned}$$

where (4.3) and (4.4) are obtained by using the fact that $\overline{\psi_{\lambda,l}(u)} = \psi_{\lambda,l}\left(u^{-1}\right)$ for all $u \in U$.

Since
$$\int_K \psi_{\lambda,l}(u_1 k u_2)\chi_l(k) d\mu_K(k) = \psi_{\lambda,l}(u_1)\psi_{\lambda,l}(u_2), \text{ for all } u_1, u_2 \in U,$$
we infer that

$$\begin{aligned}
&\mathrm{Tr}(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda)) = \int_K \cdots \int_K \psi_{\lambda,l}\left(a_r^{-1} k_r^{-1} \cdots k_2^{-1} a_1^{-1}\right)\chi_l(k_2 \ldots k_r)^{-1} d\mu_K(k_2)\cdots d\mu_K(k_r)\\
&\quad = \psi_{\lambda,l}(a_1^{-1})\psi_{\lambda,l}(a_r^{-1})\\
&\qquad \times \int_K \cdots \int_K \psi_{\lambda,l}\left(a_{r-1}^{-1} k_{r-1}^{-1} \cdots k_3^{-1} a_2^{-1}\right)\chi_l(k_3 \ldots k_{r-1})^{-1} d\mu_K(k_3)\cdots d\mu_K(k_{r-1})\\
&\qquad \vdots\\
&\quad = \prod_{i=1}^{r} \psi_{\lambda,l}\left(a_i^{-1}\right).
\end{aligned}$$

Hence the Proposition.

□

Let Δ denote the Casimir operator, $\langle\cdot,\cdot\rangle$ the inner product on $\mathfrak{a}^*$ induced by the Killing form of $\mathfrak{u}$, and let
$$\kappa_\lambda = \langle \lambda, \lambda + 2\rho\rangle$$
be the Casimir constant, where 2ρ is the sum of all positive roots and $\lambda \in \mathfrak{a}^*$.

Denote by $H^s(U)$ the Sobolev space of functions in $L^2(U)$ whose weak derivatives up to order s are in $L^2(U)$, and let $\|\cdot\|_{H^s(U)}$ denote the corresponding Sobolev norm. For more details, see [**3**].

Proposition 4.5. *With the notation above, we have*

$$\left\| f_{a_1,\ldots,a_r,\chi_l} \right\|^2_{H^s(U)} = \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda \left(1+\kappa_\lambda\right)^s \prod_{i=1}^{r} \left|\psi_{\lambda,l}\left(a_i\right)\right|^2 .$$

Proof. Note first that

$$\begin{aligned}
\left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda), \mathcal{F}(f_{a_1,\ldots,a_r,\chi_l})(\lambda)\right)_{HS} &= \sum_{i=1}^{d_\lambda} \left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_i^\lambda, \mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_i^\lambda\right)_\lambda \\
&= \left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_1^\lambda, \mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_1^\lambda\right)_\lambda \\
&= \left\| \mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_1^\lambda \right\|_\lambda^2 \\
&= \left\| \left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda) e_1^\lambda, e_1^\lambda\right)_\lambda e_1^\lambda \right\|_\lambda^2 \\
&= \left|\mathrm{Tr}\left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda)\right)\right|^2 . \qquad (4.5)
\end{aligned}$$

Equation (4.5) is a consequence of Lemma 4.1 and Lemma 4.3.

Combining (4.2) and (4.5), we obtain

$$\begin{aligned}
\left\| f_{a_1,\ldots,a_r,\chi_l} \right\|^2_{H^s(U)} &= \left\| (I-\Delta)^{\frac{s}{2}} f_{a_1,\ldots,a_r,\chi_l} \right\|^2_{L^2(U)} \\
&= \left((I-\Delta)^s f_{a_1,\ldots,a_r,\chi_l}, f_{a_1,\ldots,a_r,\chi_l}\right)_{L^2(U)} \\
&= \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda \left(\mathcal{F}\left((I-\Delta)^s f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda), \mathcal{F}(f_{a_1,\ldots,a_r,\chi_l})(\lambda)\right)_{HS} \\
&= \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda \left((1+\kappa_\lambda)^s \mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda), \mathcal{F}(f_{a_1,\ldots,a_r,\chi_l})(\lambda)\right)_{HS} \\
&= \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda (1+\kappa_\lambda)^s \left|\mathrm{Tr}\left(\mathcal{F}\left(f_{a_1,\ldots,a_r,\chi_l}\right)(\lambda)\right)\right|^2 \\
&= \sum_{\lambda \in \Lambda_l^+(U)} d_\lambda (1+\kappa_\lambda)^s \prod_{i=1}^{r} \left|\psi_{\lambda,l}(a_i)\right|^2 .
\end{aligned}$$

□

5. Restricted Roots on Complex Grassmannians

Let p and q be two integers such that $p \geq q \geq 2,\ n = p+q$, and let

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

where I_n is the $n \times n$-identity matrix. Consider the non-compact Lie group

$$G = SU(p,q) = \left\{ g \in SL(p+q,\mathbb{C}) \mid g^* I_{p,q} g = I_{p,q} \right\}.$$

Let $\mathfrak{g} = \mathfrak{su}(p,q)$ denote the Lie algebra of $SU(p,q)$, then

$$\mathfrak{su}\,(p,q) = \left\{ A \in M_{p+q}(\mathbb{C}) \mid A^* I_{p,q} + I_{p,q} A = 0, \mathrm{Tr}\,(A) = 0 \right\}.$$

Let $\mathfrak{su}(p,q) = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{su}(p,q)$, with

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{u}\,(p)\,,\ B \in \mathfrak{u}\,(q)\ \text{ and }\ \mathrm{Tr}\,(A) + \mathrm{Tr}\,(B) = 0 \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ \overline{Z}^T & 0 \end{pmatrix} \mid Z \in M_{p,q}(\mathbb{C}) \right\}.$$

Let $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ be the compact real form of $\mathfrak{sl}(p+q,\mathbb{C})$, the complexification of $\mathfrak{su}(p,q)$. Thus $U = SU(p+q)$ is the lie group with lie algebra $\mathfrak{u}$.
Take $K = S(U(p) \times U(q))$. Then the Lie algebra of K is $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. For $T = (t_1, t_2, \ldots, t_q) \in \mathbb{R}^q$ we choose $\mathfrak{a}$ to be all $(p+q) \times (p+q)$-matrices of the form

$$H_T = \left(\begin{array}{c:c:c} 0 & 0 & \begin{matrix} & & t_1 \\ & \iddots & \\ t_q & & \end{matrix} \\ \hdashline 0 & 0 & 0 \\ \hdashline \begin{matrix} & & t_q \\ & \iddots & \\ t_1 & & \end{matrix} & 0 & 0 \end{array} \right).$$

Therefore we can identify $\mathfrak{a}$ and $\mathfrak{a}^*$, the dual of $\mathfrak{a}$, with $\mathbb{R}^q$. Let $\alpha_i \in \mathfrak{a}^*$ be such that

$$\alpha_i \left(H_{(t_1,\ldots,t_q)} \right) = t_i.$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, be the set of restricted roots, so we have

$$\Sigma = \{ \pm\alpha_i, \pm 2\alpha_i, (1 \le i \le q), \text{ and } \pm(\alpha_i \pm \alpha_j), (1 \le i < j \le q) \},$$

with multiplicities

$$m_{\alpha_i} = 2k, \ \ m_{2\alpha_i} = 1, \text{ and } m_{\alpha_i \pm \alpha_j} = 2,$$

where $k = p - q$. Let C^+ be a Weyl chamber in $\mathfrak{a}$ such that

$$\mathsf{C}^+ = \left\{ H_{(t_1,\ldots,t_q)} \in \mathfrak{a} \,\middle|\, t_1 > t_2 > \cdots > t_q > 0 \right\},$$

so the corresponding system of positive restricted roots Σ^+ consists of

$$\alpha_i, 2\alpha_i, (1 \le i \le q), \text{ and } (\alpha_i \pm \alpha_j), (1 \le i < j \le q).$$

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Then

$$\begin{aligned} \rho &= \frac{1}{2} \left(\sum_{i=1}^{q} 2k\alpha_i + \sum_{i=1}^{q} 2\alpha_i + \sum_{i=1}^{q} 4(q-i)\alpha_i \right) \\ &= \sum_{i=1}^{q} (k+1+2(q-i))\alpha_i. \end{aligned}$$

By ([**11**],Proposition 7.1, Theorem 7.2; [**8**], Theorem 3.1), we have

$$\Lambda_l^+(U) = \left\{ \lambda = \sum_{i=1}^{q} \lambda_i \alpha_i \in \mathfrak{a}^* \,\middle|\, \lambda_i - \lambda_j \in 2\mathbb{Z}^+ (1 \le i < j \le q),\ \lambda_1 \in |l| + 2\mathbb{Z}^+ \right\}.$$

Hence we have

$$\Lambda_l^+(U) = \{\lambda = (2m_1 + |l|, 2m_2 + |l|, \ldots, 2m_q + |l|) \mid m_i \in \mathbb{Z}, m_1 \ge m_2 \ge \cdots \ge m_q \ge 0\}.$$

An explicit formula for the χ_l-spherical functions on the complex Grassmannians $(SU(p,q)/S(U(p) \times U(q)))$ was obtained in [**1**]. More precisely, let

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}\right)$$

be the Jacobi polynomial of degree n, where ${}_2F_1(\,\cdot\,,\,\cdot\,,\,\cdot\,;\,\cdot\,)$ is the Gauss hyper-geometric function, and let

$$\tilde{P}_{n,l}(\cos(2x)) = \frac{P_n^{(k,|l|)}(\cos(2x))}{P_n^{(k,|l|)}(1)} = {}_2F_1\left(n+k+|l|+1, -n, k+1; \sin^2(x)\right).$$

Then we have

Theorem 5.1 ([**1**])**.** *Let $\lambda \in \Lambda_l^+(U)$ such that*

$$\lambda = (2m_1 + |l|, 2m_2 + |l|, \ldots, 2m_q + |l|), m_i \in \mathbb{Z}, m_1 \ge m_2 \ge \cdots \ge m_q \ge 0.$$

Then the χ_l-spherical function ψ_λ on $U = SU(p+q)$ is given by

$$\psi_{\lambda,l}\left(\exp\left(\sqrt{-1}H_{(t_1,\ldots,t_q)}\right)\right) = \frac{C \det\left[\tilde{P}_{n_i,l}(\cos(2t_j))\right]_{i,j} \prod_{i=1}^{q} \cos^{|l|}(t_i)}{\prod_{1\le i<j\le q} (c(n_i) - c(n_j)) \prod_{1\le i<j\le q} (\cos(2t_i) - \cos(2t_j))},$$

where $H_{(t_1,\ldots,t_q)} \in \mathfrak{a}$, $k = p - q$, $n_i = m_i + q - i$, $c(n_i) = n_i(n_i + |l| + k + 1)$ and

$$C = 2^{\frac{1}{2}q(q-1)} \prod_{j=1}^{q-1} \left[(k+j)^{q-j} j!\right].$$

Note that the χ_l-spherical function $\psi_{\lambda,l}$ in Theorem 5.1 can be defined for all elements of U by means of the decomposition

$$U \;=\; K \exp\big(i\,\mathfrak{a}\big)\, K \quad \text{(see Theorem 7.39 in [9])}.$$

In other words, since $\psi_{\lambda,l}$ is χ_l-bi-invariant, for any $u \in U$,

$$u \;=\; k_1 \exp\big(i\,X\big)\, k_2, \qquad \text{for some } k_1,\ k_2 \in K, \text{ and } X \in \mathfrak{a},$$

we obtain

$$\psi_{\lambda,l}(u) \;=\; \chi_l(k_1k_2)^{-1}\, \psi_{\lambda,l}\big(\exp(i\,X)\big).$$

Corollary 5.2. *Let $\lambda \in \Lambda_l^+(U)$ such that*

$$\lambda = (2m_1 + |l|, 2m_2 + |l|, \ldots, 2m_q + |l|), m_i \in \mathbb{Z}, m_1 \ge m_2 \ge \cdots \ge m_q \ge 0,$$

and let

$$u \;=\; k_1 \exp\big(i\,H_{(t_1,\ldots,t_q)}\big)\, k_2,$$

where $k_1,\ k_2 \in K$, and $H_{(t_1,\ldots,t_q)} \in \mathfrak{a}$.

It follows for sufficiently large n_j we have

$$|\psi_{\lambda,l}(u)| = \left|\psi_{\lambda,l}\left(\exp\left(\sqrt{-1}H_{(t_1,\dots,t_q)}\right)\right)\right| \le \begin{cases} \frac{C(l,t_1,\dots,t_q)}{\prod_{j=1}^{q} n_j^{\frac{1}{2}(2p-q)}} & \text{if } m_q > 0, \\ \frac{C(l,t_1,\dots,t_q)}{\prod_{j=1}^{q-1} n_j^{\frac{1}{2}(2p-q+3)}} & \text{if } m_q = 0, \end{cases}$$

where $H_{(t_1,\dots,t_q)} \in \mathfrak{a}$ and $C(l,t_1,\dots,t_q)$ is a constant depending on $t_1,\dots,t_q$ and l.

Proof. By Theorem 5.1 and a similar argument as in [**2**]. □

6. Proof of the Main Theorem

Lemma 6.1. *Let (π_λ, E_λ) be an irreducible χ_l-spherical representation of $U = SU(p+q)$ with highest weight $\lambda = (2m_1+|l|,\dots,2m_q+|l|)$ and $d_\lambda = \dim(E_\lambda)$. Then*

$$d_\lambda = \begin{cases} \prod_{i=1}^{q} \frac{\varphi\left(2m_i+|l|+k+1+2(q-i);\, \frac{1}{2}(k-1)\right)}{\varphi\left(k+1+2(q-i);\, \frac{1}{2}(k-1)\right)} \prod_{i=1}^{q} \frac{2m_i+|l|+k+1+2(q-i)}{k+1+2(q-i)} \\ \quad \times \prod_{1\le i<j\le q} \left(\frac{m_i+m_j+|l|+k+2q-(i+j)}{k+2q-(i+j)}\right)^2 \prod_{1\le i<j\le q}\left(\frac{m_i-m_j+j-i}{j-i}\right)^2 & \text{if } p \neq q, \\ \prod_{i=1}^{q} \frac{2m_i+|l|+1+2(q-i)}{1+2(q-i)} \prod_{1\le i<j\le q}\left(\frac{m_i+m_j+|l|+1+2q-(i+j)}{1+2q-(i+j)}\right)^2 \\ \quad \times \prod_{1\le i<j\le q}\left(\frac{m_i-m_j+j-i}{j-i}\right)^2 & \text{if } p = q. \end{cases}$$

Proof. Let

$$\Phi(x,y;m) = \frac{\varphi(x+y;\, m)}{\varphi(y;\, m)},$$

with

$$\varphi(x;t) = (x-t)(x-t+1)\cdots(x+t)$$

Let

$$W(x,y,m,1) = \Phi(x,y;0)\left[\Phi\left(x,y;\frac{1}{4}m-\frac{1}{2}\right)\right]^2,$$

$$W(x,y,m,0) = \begin{cases} \Phi(x,y;0), & \text{if } m=1, \\ \Phi(x,y;0)^2 & \text{if } m=2. \end{cases}$$

Then by [**7**, Theorem 2.1], we have

$$d_\lambda = \prod_{\alpha\in\Sigma_0^+} W\Big(\langle\lambda,\alpha\rangle, \langle\rho,\alpha\rangle; m_\alpha, m_{2\alpha}\Big),$$

where Σ_0^+ is the set of the indivisible positive roots. For $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\mu = (\mu_1, \ldots, \mu_q) \in \mathfrak{a}^*$, and $n = p + q$, we have

$$\langle \lambda, \mu \rangle = 4n \sum_{i=1}^{q} \lambda_i \mu_i.$$

Therefore for $p \neq q$ we have

$$\begin{aligned}
d_\lambda &= \prod_{\alpha \in \Sigma_0^+} W\Big(\langle \lambda, \alpha \rangle, \langle \rho, \alpha \rangle; m_\alpha, m_{2\alpha}\Big) \\
&= \prod_{i=1}^{q} W\Big(4n(2m_i + |l|), 4n[k + 1 + 2(q - i)]; 2k, 1\Big) \\
&\quad \times \prod_{1 \leq i < j \leq q} W\Big(8n(m_i + m_j + |l|), 8n(k + 1 + 2q - (i + j)); 2, 0\Big) \prod_{1 \leq i < j \leq q} W\Big(8n(m_i - m_j), 8n(j - i); 2, 0\Big) \\
&= \prod_{i=1}^{q} \Phi\left(4n(2m_i + |l|), 4n[k + 1 + 2(q - i)]; \frac{1}{2}\,(k - 1)\right)^2 \prod_{i=1}^{q} \Phi\Big(4n(2m_i + |l|), 4n[k + 1 + 2(q - i)]\Big) \\
&\quad \times \prod_{1 \leq i < j \leq q} \Phi\Big(8n(m_i + m_j + |l|), 8n(k + 2q - (i + j))\Big)^2 \prod_{1 \leq i < j \leq q} \Phi\Big(8n(m_i - m_j), 8n(j - i)\Big)^2 \\
&= \prod_{i=1}^{q} \frac{\varphi\Big(2m_i + |l| + k + 1 + 2(q - i); \frac{1}{2}(k - 1)\Big)}{\varphi\Big(k + 1 + 2(q - i); \frac{1}{2}(k - 1)\Big)} \prod_{i=1}^{q} \frac{2m_i + |l| + k + 1 + 2(q - i)}{k + 1 + 2(q - i)} \\
&\quad \times \prod_{1 \leq i < j \leq q} \left(\frac{m_i + m_j + |l| + k + 2q - (i + j)}{k + 2q - (i + j)}\right)^2 \prod_{1 \leq i < j \leq q} \left(\frac{m_i - m_j + j - i}{j - i}\right)^2.
\end{aligned}$$

A similar argument can be applied for the case $p = q$. □

Corollary 6.2. *Let*

$$\lambda = \big(2m_1 + |l|,\ 2m_2 + |l|,\ \ldots,\ 2m_q + |l|\big) \ \in\ \Lambda_l^+(U).$$

Then

$$d_\lambda \ \leq\ \big(2m_1 + |l| + 1\big)^{q(2p-1)} \ \leq\ C\,(l, p, q)\ n_1^{q(2p-1)},$$

where $n_1 = m_1 + q - 1$ *and* $C\,(l, p, q)$ *is a positive constant depending on* l, p, *and* q.

Proof. Suppose that $p > q$. Then

$$\begin{aligned}
\frac{\varphi\Big(2m_i + |l| + k + 1 + 2(q - i); \frac{1}{2}(k - 1)\Big)}{\varphi\Big(k + 1 + 2(q - i); \frac{1}{2}(k - 1)\Big)} &= \prod_{j=0}^{k-1} \frac{2m_i + |l| + k + 1 + 2(q - i) - \frac{1}{2}(k - 1) + j}{k + 1 + 2(q - i) - \frac{1}{2}(k - 1) + j} \\
&= \prod_{j=0}^{k-1} \left(\frac{2m_i + |l|}{k + 1 + 2(q - i) - \frac{1}{2}(k - 1) + j} + 1\right) \\
&\leq (2m_1 + |l| + 1)^k,
\end{aligned}$$

where $k + 1 + 2(q - i) - \frac{1}{2}(k - 1) + j \geq k + 1 + 2(q - i) - \frac{1}{2}(k - 1) > 1$.

It follows that

$$\prod_{i=1}^{q} \frac{\varphi\Big(2m_i+|l|+k+1+2(q-i);\frac{1}{2}(k-1)\Big)}{\varphi\Big(k+1+2(q-i);\frac{1}{2}(k-1)\Big)} \prod_{i=1}^{q} \frac{2m_i+|l|+k+1+2(q-i)}{k+1+2(q-i)}$$

$$\times \prod_{1\le i<j\le q}\left(\frac{m_i+m_j+|l|+k+2q-(i+j)}{k+2q-(i+j)}\right)^2 \prod_{1\le i<j\le q}\left(\frac{m_i-m_j+j-i}{j-i}\right)^2$$

$$\le (2m_1+|l|+1)^{\,qk+q+q(q-1)+q(q-1)}$$

$$= (2m_1+|l|+1)^{\,q(p-q)+q+2q(q-1)}$$

$$\le (2m_1+|l|+1)^{\,q(2p-1)}.$$

A similar argument applies when $p=q$. □

Lemma 6.3. *Let $\lambda=(2m_1+|l|,\dots,2m_q+|l|)\in\Lambda_l^+(U)$, with $\|\lambda\|\ge\|\rho\|$ then*

$$\kappa_\lambda \le\ C\,(l)\ n_1^2.$$

for some positive constant $C\,(l)$ depending on l.

Proof. Let $\lambda=(2m_1+|l|,\dots,2m_q+|l|)\in\Lambda_l^+(U)$, by the Cauchy-Schwarz inequality, for $\lambda=(2m_1+|l|,\dots,2m_q+|l|)\in\Lambda_l^+(U)$,

$$\begin{aligned}\kappa_\lambda &= \langle\lambda+2\rho,\lambda\rangle\\ &\le \|\lambda+2\rho\|\ \|\lambda\|\\ &\le \|\lambda\|^2+2\,\|\rho\|\ \|\lambda\|\\ &\le 3\|\lambda\|^2.\end{aligned}$$

Since all norms on a finite-dimensional vector space are equivalent,

$$\kappa_\lambda \le\ 3\|\lambda\|^2\ \le\ C_1\left(2m_1+|l|\right)^2\ \le\ C_1\left(2n_1+|l|\right)^2\ \le\ C\,(l)\ n_1^2,$$

for some positive constants C_1 and $C\,(l)$. □

Proposition 6.4. *With the above notation, we have*

$$\|f_{a_1,\dots,a_r,\chi_l}\|^2_{H^s(U)} \le \sum_{\substack{\lambda\in\Lambda_l^+(U)\\ \|\lambda\|<\|\rho\|}} d_\lambda\left(1+\kappa_\lambda\right)^s\prod_{i=1}^{r}\left|\psi_{\lambda,l}(a_i)\right|^2$$

$$+\,C(l,a_1,\dots,a_r,p,q)\left[\sum_{n=1}^{\infty}\frac{1}{n^{\,r(2p-q)-2s-q(2p-1)}}+\sum_{n=1}^{\infty}\frac{1}{n^{\,r(2p-q+3)-2s-q(2p-1)}}\right],$$

where $C(l,a_1,\dots,a_r,p,q)$ is a positive constant depending on l, $a_1,\dots,a_r$, p and q.

Proof. Using Corollary 6.2, Corollary 5.2, Lemma 6.3 and Proposition 4.5, we get

$$
\begin{aligned}
&\|f_{a_1,\ldots,a_r,\chi_l}\|^2_{H^s(U)} - \sum_{\substack{\lambda\in\Lambda_l^+(U)\\ \|\lambda\|<\|\rho\|}} d_\lambda\,(1+\kappa_\lambda)^s\,\prod_{i=1}^{r}|\psi_{\lambda,l}(a_i)|^2\\
&= \sum_{\lambda\in\Lambda_l^+(U)} d_\lambda\,(1+\kappa_\lambda)^s\,\prod_{i=1}^{r}|\psi_{\lambda,l}(a_i)|^2 - \sum_{\substack{\lambda\in\Lambda_l^+(U)\\ \|\lambda\|<\|\rho\|}} d_\lambda\,(1+\kappa_\lambda)^s\,\prod_{i=1}^{r}|\psi_{\lambda,l}(a_i)|^2\\
&= \sum_{\substack{\lambda\in\Lambda_l^+(U)\\ \|\lambda\|\geq\|\rho\|}} d_\lambda\,(1+\kappa_\lambda)^s\,\prod_{i=1}^{r}|\psi_{\lambda,l}(a_i)|^2\\
&\leq C_1(l,a_1,\ldots,a_r,p,q)\left[\sum_{n_1>n_2>\cdots>n_q\geq1}\frac{n_1^{2s+q(2p-1)}}{\prod_{j=1}^{q}n_j^{r(2p-q)}} + \sum_{n_1>n_2>\cdots>n_{q-1}\geq1}\frac{n_1^{2s+q(2p-1)}}{\prod_{j=2}^{q-1}n_j^{r(2p-q+3)}}\right]\\
&\leq C_1(l,a_1,\ldots,a_r,p,q)\left[\sum_{n_1>n_2>\cdots>n_q\geq1}\frac{1}{n_1^{r(2p-q)-2s-q(2p-1)}\,\prod_{j=2}^{q}n_j^{r(2p-q)}}\right.\\
&\qquad\left.+\sum_{n_1>n_2>\cdots>n_{q-1}\geq1}\frac{1}{n_1^{r(2p-q+3)-2s-q(2p-1)}\,\prod_{j=2}^{q-1}n_j^{r(2p-q+3)}}\right]\\
&\leq C_1(l,a_1,\ldots,a_r,p,q)\left[\left(\sum_{n_1=1}^{\infty}\frac{1}{n_1^{r(2p-q)-2s-q(2p-1)}}\right)\left(\sum_{n_2=1}^{\infty}\frac{1}{n_2^{r(2p-q)}}\right)\cdots\left(\sum_{n_q=1}^{\infty}\frac{1}{n_q^{r(2p-q)}}\right)\right.\\
&\qquad\left.+\left(\sum_{n_1=1}^{\infty}\frac{1}{n_1^{r(2p-q+3)-2s-q(2p-1)}}\right)\left(\sum_{n_2=1}^{\infty}\frac{1}{n_2^{r(2p-q)}}\right)\cdots\left(\sum_{n_{q-1}=1}^{\infty}\frac{1}{n_{q-1}^{r(2p-q)}}\right)\right],\\
&\leq C_1(l,a_1,\ldots,a_r,p,q)\left[\sum_{k=1}^{\infty}\frac{1}{k^{\,r(2p-q)}}\right]^{q-2}\left[\sum_{n=1}^{\infty}\frac{1}{n^{\,r(2p-q)-2s-q(2p-1)}}\right.\\
&\qquad\qquad\left.+\sum_{n=1}^{\infty}\frac{1}{n^{\,r(2p-q+3)-2s-q(2p-1)}}\right],
\end{aligned}
$$

for some positive constant $C_1(l,a_1,\ldots,a_r,p,q)$. Since $2p-q\geq p\geq 2$, we have $r(2p-q)\geq 2$. Thus $\sum_{k=1}^{\infty}k^{-\,r(2p-q)}$ converges. Hence the proposition. □

Corollary 6.5. *If*

$$r \;>\; \frac{1+2s+q\,(2p-1)}{2p-q},$$

then the function $f_{a_1,\ldots,a_r,\chi_l}$ is in $H^s(U)$. In other words,

$$\left\|f_{a_1,\ldots,a_r,\chi_l}\right\|_{H^s(U)} \;<\; \infty.$$

Proof. Observe that

$$\sum_{\substack{\lambda\in\Lambda_l^+(U)\\ \|\lambda\|<\|\rho\|}} d_\lambda\,\left(1+\kappa_\lambda\right)^s\,\prod_{i=1}^{r}\left|\psi_{\lambda,l}(a_i)\right|^2$$

is a finite sum. Also

$$\sum_{n=1}\frac{1}{n^{r(2p-q)-2s-q(2p-1)}}$$

converges whenever

$$r \ > \ \frac{1+2s+q(2p-1)}{2p-q}.$$

Moreover

$$\sum_{n=1}^{\infty} \frac{1}{n^{r(2p-q+3)-2s-q(2p-1)}}$$

converges if

$$r \ > \ \frac{1+2s+q(2p-1)}{2p-q+3}.$$

Hence by Proposition 6.4 the result follows. □

Let ν be a positive integer and define

$$C\big(p,q,\nu\big) \ := \ \max\Big\{ \Big\lfloor \tfrac{(p+q)^2+2\nu+q(2p-1)}{2p-q} \Big\rfloor + 1, \ 2pq \Big\}.$$

Theorem 6.6. *Let ν be a positive integer. If*

$$r \ \geq \ C\big(p,q,\nu\big),$$

then

$$\frac{d\big(\mu_{a_1,\chi_l} * \cdots * \mu_{a_r,\chi_l}\big)}{d\mu_U} \ \in \ \mathcal{C}^{\,\nu}\big(SU(p+q)\big).$$

Proof. Observe first that

$$\dim\Big(SU(n)/S\big(U(p)\times U(q)\big)\Big) \ = \ 2pq.$$

Thus by Theorem 3.4, $\mu_{a_1,\ldots,a_r,\,\chi_l}$ is absolutely continuous with respect to the Haar measure μ_U of U, i.e., $f_{a_1,\ldots,a_r,\chi_l}\, d\,\mu_U = d\,\mu_{a_1,\ldots,a_r,\,\chi_l}$.

Let $r \geq C(p,q,\nu)$ be a positive integer. By definition of $C(p,q,\nu)$, we can choose $\epsilon > 0$ such that

$$r \ > \ \frac{(p+q)^2+2\nu+q(2p-1)+2\epsilon}{2p-q}.$$

Then, by Corollary 6.5 (taking $s = \nu \ + \ \dfrac{(p+q)^2-1}{2} \ + \ \epsilon$), we have

$$f_{a_1,\ldots,a_r,\chi_l} \ \in \ H^s\big(SU(p+q)\big).$$

Therefore, the Sobolev Embedding Theorem implies

$$H^s\big(SU(p+q)\big) \ \subseteq \ \mathcal{C}^{\nu}\big(SU(p+q)\big).$$

Hence the result follows. □

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