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LEFT INVERSION IN A DUAL BANACH ALGEBRA ENDOWED WITH AN ARENS PRODUCT AND THE JACOBSON RADICAL

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Abstract. With some suitable Arens product, the dual space of the C* algebra of left uniformly continous complex valued functions on a locally compact group G admits a Banach algebra structure. The nature of the left invertible elements is important for its connection with the Jacobson radical. Our aim in this article is to determine conditions of left invertibility and some of their consequences.

1. The Spaces LUC(G), RUC(G), UC(G), And Their Duals

1.1. Spaces of uniformly continuous functions. Throughout this article G will denote a locally compact group with unit element e. Let LUC(G) (resp. RUC(G)) be the B*-subalgebras of $\mathrm{C}_b(\mathrm{G})$ of the left (resp. right) uniformly continuous functions on G, i.e. the functions $x \in \mathrm{C}_b(\mathrm{G})$ such that the mapping $g \in G \xrightarrow{l(x)} {}_g x \in \mathrm{C}_b(G)$ (resp. $g \in G \xrightarrow{r(x)} x_g \in \mathrm{C}_b(G)$) is continuous, with ${}_g x(h) = x(gh)$ (resp. $x_g(h) = x(hg)$) if $g, h \in G$. We shall denote UC(G) the space of uniformly continuous functions on G, i.e. $\mathrm{UC(G)} = \mathrm{LUC(G)} \cap \mathrm{RUC(G)}$. In particular, the mapping $r_L : G \to \mathcal{B}(\mathrm{LUC(G)})$ (resp. $l_L : G \to \mathcal{B}(\mathrm{LUC(G)})$) is given such that $r_L(g)(x) = x_g$ (resp. $l_L(g)(x) =_g x$) is a well defined representation (resp. anti-representation) of G into LUC(G). With obvious changes we have similar situations with mappings $G \xrightarrow{l_R, r_R} \mathcal{B}(\mathrm{RUC(G)})$. Additionally, given $x \in \mathrm{C}_b(\mathrm{G})$ and $g \in G$ let $\iota(x)(g) = x(g^{-1})$. If we write $\iota_{LR} = \iota \mid_{\mathrm{LUC(G)}}$ and $\iota_{RL} = \iota \mid_{\mathrm{RUC(G)}}$ then $\mathrm{Im}(\iota_{LR}) = \mathrm{RUC(G)})$, $\iota_{LR} \in \mathcal{B}(\mathrm{LUC(G)}, \mathrm{RUC(G)})$ and $\iota_{LR} = \iota_{RL}^{-1}$.

1.2. The Banach algebras LUC(G)* and RUC(G)*. [**3**] [**8**] The dual spaces $\mathrm{LUC}(G)^*$ and $\mathrm{RUC(G)}^*$ admit Banach algebra structures. For $m, n \in \mathrm{LUC}(G)^*$, $p, q \in \mathrm{RUC(G)}^*$, $x \in \mathrm{LUC}(G)$, $y \in \mathrm{RUC(G)}$ and $g \in G$, let

$$\langle x, m\Box n\rangle = \langle \rho_L(n)(x), m\rangle \text{ and } \langle y, p\Diamond q\rangle = \langle \rho_R(p)(y), q\rangle,$$

where

$$\rho_L : \mathrm{LUC(G)}^* \to \mathcal{B}(\mathrm{LUC(G)}),\ \rho_L(n)(x)(g) = \langle {}_g x, n\rangle,$$
$$\rho_R : \mathrm{RUC(G)}^* \to \mathcal{B}(\mathrm{RUC(G)}),\ \rho_R(p)(y)(g) = \langle y_g, p\rangle.$$

For brevity we shall also write $\rho_L(n) = n_l$ and $\rho_R(p) = p_r$ for each $n \in \mathrm{LUC(G)}^*$ and $p \in \mathrm{UUC(G)}^*$. We now list some properties whose proofs are straightforward:

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(i) The mappings ρ_L and ρ_R are well defined contractive linear operators.
(ii) $m\Box n \in \mathrm{LUC(G)}^*$.
(iii) $p\Diamond q \in \mathrm{RUC(G)}^*$.
(iv) $(\mathrm{LUC(G)}^*, \Box)$ and $(\mathrm{RUC(G)}^*, \Diamond)$ become unital associative Banach algebras.
(v) ρ_L is a faithful continuous representation of $\mathrm{LUC(G)}^*$ on LUC(G) and

$$\mathrm{Im}(\rho_L) = \{l_L(g) : g \in G\}^c.$$

(vi) ρ_R is a bounded anti-representation of $\mathrm{RUC(G)}^*$ on RUC(G) and

$$\mathrm{Im}(\rho_R) = \{r_R(g) : g \in G\}^c.$$

(vii) $\iota_{LR}^*(p\Diamond q) = \iota_{LR}^*(q)\Box\iota_{LR}^*(p)$ for all $p, q \in \mathrm{RUC(G)}^*$.
(viii) The involution $x \to x^*$ in UC(G), $x^*(g) = x(g^{-1})^-$, induces an isometric involution $m \to m^*$ in $\mathrm{UC(G)}^*$ so that $\langle x, m^*\rangle = \langle x^*, m\rangle^-$. As $(m\Box n)^* = n^*\Diamond m^*$ for all $m, n \in \mathrm{UC(G)}^*$, $\mathrm{UC(G)}^*$ is a Banach *-algebra if G is abelian.

1.3. Connections with the measure algebra. The group measure algebra $\mathrm{M}(G)$ of complex bounded regular Borel measures on G is naturally embedded in $\mathrm{LUC}(G)^*$ if, for $\mu \in \mathrm{M}(G)$ and $x \in \mathrm{LUC}(G)$, we set $\langle x, \hat{\mu}\rangle = \int_G x d\mu$. It is straightforward to see that $\mu \to \hat{\mu}$ is a monomorphism of M(G) into $\mathrm{LUC(G)}^*$. Separately, the subspace $\mathrm{C}_{00}(G)$ of $\mathrm{B}_b(G)$ of functions with compact support is clearly contained in $\mathrm{LUC}(G)$. As a consequence of Urysohn's lemma $\mathrm{C}_0(G) \subseteq \mathrm{LUC}(G)$, where $\mathrm{C}_0(G) = \mathrm{C}_{00}(G)^-$ is the closed subspace of $\mathrm{C}_b(G)$ of functions which vanish at infinity. In particular, if G is discrete then

$$\mathrm{LUC(G)} = \mathrm{C}_b(\mathrm{G}) = \mathrm{l}^\infty(\mathrm{G}) = \mathrm{C}_0(\mathrm{G})^-,$$

and $\mathrm{LUC(G)}^*$ realizes as the space of bounded finitely additive complex measures on G [**9**]. In the general case $\mathrm{LUC}(G)^* = \mathrm{C}_0(G)^\perp \oplus_1 \mathrm{M(G)}^\wedge$ and $\mathrm{C}_0(\mathrm{G})^\perp$ is a closed ideal in $\mathrm{LUC(G)}^\perp$ (cf. [**7**], Lemma 1.1). In particular, M(G) is isometrically embedded onto a closed subalgebra of $\mathrm{LUC(G)}^*$.

Proposition 1.1. *The Banach algebra ($LUC(G)^*, \Box$) is never abelian.*

Proof. Let $g \in G$, $n \in \mathrm{LUC(G)}^*$ and $x \in \mathrm{LUC(G)}$. If $(\mathrm{LUC(G)}^*, \Box)$ were abelian we could write

$$\begin{aligned} 0 &= \langle x, \hat{\delta}_g\Box n - n\Box\hat{\delta}_g\rangle \\ &= \langle \rho_L(n)(x), \hat{\delta}_g\rangle - \langle \rho_L(\hat{\delta}_g)(x), n\rangle \\ &= \langle {}_gx - x_g, n\rangle. \end{aligned}$$

As n is arbitrary we infer that ${}_gx = x_g$, and as g is arbitrary then $x(gh) = x(hg)$ for all $g, h \in G$. By Urysohn's lemma we infer that G is abelian. Hence LUC(G) = UC(G) and $\mathrm{UC(G)}^*$ becomes abelian. However, $\mathrm{UC(G)}^*$ is non-abelian for any locally compact abelian group G (cf. [**13**], Remark 1). $\Box$

1.4. Radicals of normed algebras. The Jacobson radical of any associative algebra is the intersection of their maximal modular ideals. In the case of a complex Banach algebra $\mathcal{U}$ its Jacobson radical $\mathrm{J}(\mathcal{U})$ is the totality of elements $\mathfrak{a} \in \mathcal{U}$ such that $(\mathfrak{a}\mathfrak{b})^n \to 0_{\mathfrak{U}}$ (or $(\mathfrak{b}\mathfrak{a})^n \to 0_{\mathfrak{U}}$) for every $\mathfrak{b} \in \mathfrak{U}$ (cf. [**10**], Th. 15).
In the case of normed algebras it is natural to consider topologically irreducible representations, i.e. continuous homomorphisms of the algebra into algebras of bounded operators on Banach spaces without non-trivial closed invariant subspaces.

In this setting, given a complex normed algebra $\mathcal{U}$, its topologically irreducible radical $\mathrm{T}(\mathcal{U})$ is defined as the intersection of the kernels of all the continuous topologically irreducible representations of $\mathcal{U}$. Plainly, any strictly irreducible representation is topologically irreducible. However, there exists topologically irreducible representations which are not strictly irreducible. Hence, the topologically irreducible radical can be strictly smaller than the Jacobson radical (cf. [**5**], Example 3.1). It is worth mentioning that both radicals coincide for C* algebras [**11**]. Even if G is abelian this fact does not apply to UC(G)*, since this *-algebra is not of type C*.

1.5. The radical of LUC(G)*. The Banach algebra LUC(G)* is known to not be semisimple (cf. [**1**], Propositions 3 and 4). However, the precise determination of its radical seems to be elusive. The structure of some of their maximal ideals is known. Indeed, any maximal ideal M of LUC(G)* which contains $\mathrm{C_0(G)}^{\perp}$ determines a maximal ideal M_c of M(G) so that $M = M_c \oplus \mathrm{C_0(G)}^{\perp}$ (cf. [**6**], Th. 2.3).

1.6. Our aims and main results. Given an associative Banach algebra $\mathfrak{U}$ with unit e its Jacobson radical $\mathrm{J}(\mathfrak{U})$ consists of the elements $a \in \mathfrak{U}$ such that $e + xa$ is left invertible for all $x \in \mathfrak{U}$ (cf. [**4**], (1.5.1), p. 69). Consequently, the characterization of the left invertible elements of LUC(G)* is relevant in connection with its Jacobson radical. We provide this characterization in Th. 2.1. In Corollary 2.4 we shall relate the Jacobson radicals of LUC(G)* and M(G). In Corollary 2.5, for a locally compact abelian group G, we shall infer that the Jacobson radical of LUC(G)* is contained in $\mathrm{C_0(G)}^{\perp}$.

2. Left Invertibility In LUC(G)*

Theorem 2.1. *If an element $m + \hat{\mu} \in LUC(G)^*$ is left invertible the following conditions hold:*
(i) There exists a positive constant c such that $\| (m+\hat{\mu})_l(x) \| \geq c \mid x(e) \mid$ for all $x \in LUC(G)$.
(ii) $\mu \in Inv_l(M(G))$.
(iii) $(m+\hat{\mu})_l^{-1}(C_0(G)) \subseteq C_0(G)$.
When G is abelian the above conditions are also sufficient for the left invertibility of $m + \hat{\mu} \in UC(G)^*$.

Proof. Let $n + \hat{\nu} \in \mathrm{LUC(G)}^*$ such that $(n+\hat{\nu})\Box(m+\hat{\mu}) = \delta_e$. Then $(\nu * \mu)^\wedge = \hat{\delta_e}$ and so $\nu * \mu = \delta_e$ and (ii) holds. Given $x \in \mathrm{LUC(G)}$ we can write

$$\begin{aligned} \mid x(e) \mid &= \mid \langle x, (n+\hat{\nu})\Box(m+\hat{\mu})\rangle \mid \\ &= \mid \langle (m+\hat{\mu})_l(x), n+\hat{\nu}\rangle \mid \\ &\leq \| n+\hat{\nu} \| \| (m+\hat{\mu})_l(x) \| . \end{aligned}$$

With $c = \| n+\hat{\nu} \|^{-1}$ we obtain (i). Now, if $x \in (m+\hat{\mu})_l^{-1}(\mathrm{C_0(G)})$ we let $z = (m+\hat{\mu})_l(x)$. It is straightforward to see that the mapping $u \in \mathrm{LUC(G)}^* \to u_l \in \mathcal{B}(\mathrm{LUC(G)})$ defines a bounded homomorphism of Banach algebras. In fact it represents the elements of LUC(G)* as bounded endomorphisms of LUC(G). Now,

$$\begin{aligned}\hat{\nu}_l(z) &= (n+\hat{\nu})_l(z)\\ &= ((n+\hat{\nu})_l \circ (m+\hat{\mu})_l)(x)\\ &= ((n+\hat{\nu})\Box(m+\hat{\mu}))_l(x)\\ &= (\hat{\delta_e})_l(x)\\ &= \mathrm{Id}_{\mathrm{LUC(G)}}(x)\\ &= x.\end{aligned}$$

Consequently $x \in \mathrm{C}_0(\mathrm{G})$ ([**8**], Lemma 19.5).
From now on let us assume that G is abelian and that $m+\hat{\mu} \in \mathrm{UC(G)}^*$ satisfies conditions (i), (ii) and (iii). Thus $\mathrm{C}_0(\mathrm{G}) \subseteq \mathrm{ran}[(m+\hat{\mu})_l]$. Let $z \in \mathrm{C}_0(\mathrm{G})$ and $\eta \in \mathrm{M(G)}$ such that $\eta * \mu = \delta_e$. Since $(\hat{\eta})_l(z) \in \mathrm{C}_0(\mathrm{G})$ and given $g \in G$, we see that

$$\begin{aligned}(m+\hat{\mu})_l[(\hat{\eta})_l(z)](g) &= (\hat{\mu})_l[(\hat{\eta})_l(z)](g)\\ &= \int_G (\hat{\eta})_l(z)(gh)d\mu(h)\\ &= \int_G\int_G z(ghk)d\eta(k)d\mu(h)\\ &= \langle {}_g z, \eta * \mu\rangle\\ &= z(g),\end{aligned}$$

i.e. $z = (m+\hat{\mu})_l[(\hat{\eta})_l(z)]$. Now, let $p : \mathrm{Im}[(m+\hat{\mu})_l] \to \mathbb{C}$ so that $\langle w, p\rangle = -\langle y, \hat{\eta}\Box m\rangle$ if $w = (m+\hat{\mu})_l(y)$ for some $y \in \mathrm{UC(G)}$. Given $y' \in \ker[(m+\hat{\mu})_l]$ we have

$$\langle y', \hat{\eta}\Box m\rangle = \langle m_l(y'), \hat{\eta}\rangle = -\langle(\hat{\mu})_l(y), \hat{\eta}\rangle = -\langle y', \hat{\delta_e}\rangle = 0.$$

Thus p is a well defined function on $\mathrm{ran}(m+\hat{\mu})_l$. Furthermore, if $w = (m+\hat{\mu})_l(y)$ belongs to $\mathrm{C}_0(\mathrm{G})$, then by (iii) $y \in \mathrm{C}_0(\mathrm{G})$. Hence $\langle w, p\rangle = -\langle m_l(y), \hat{\eta}\rangle = 0$ because $m_l(y) = 0_{\mathrm{UC(G)}}$. Clearly p is a complex linear functional and continuing with the above notation we have

$$\begin{aligned}\mid \langle y, \hat{\eta}\Box m\rangle \mid &=\mid \langle m_l(y), \hat{\eta}\rangle \mid\\ &=\mid \langle (m+\hat{\mu})_l(y), \hat{\eta}\rangle - \langle \hat{\mu}(y), \hat{\eta}\rangle \mid\\ &\leq \parallel \hat{\eta} \parallel\parallel (m+\hat{\mu})_l(y) \parallel + \mid \int_G\int_G y(gh)d\mu(g)d\eta(h) \mid\\ &=\parallel \hat{\eta} \parallel\parallel (m+\hat{\mu})_l(y) \parallel + \mid y(e) \mid\\ &\leq (\parallel \eta \parallel + c^{-1}) \parallel (m+\hat{\mu})_l(y) \parallel .\end{aligned}$$

Thus the Hahn-Banach theorem provides a linear extension $p_1 : \mathrm{UC(G)} \to \mathbb{C}$ such that $\mid \langle x, p_1\rangle \mid \leq (\parallel \eta \parallel + c^{-1}) \parallel x \parallel$ for all $x \in \mathrm{UC(G)}$, i.e. $p_1 \in \mathrm{UC(G)}^*$. Moreover, $p_1 \in \mathrm{C}_0(\mathrm{G})^{\perp}$ and clearly $(p_1+\hat{\eta})\Box(m+\hat{\mu}) = \hat{\delta_e}$. □

Corollary 2.2. *Let $\mathfrak{m} \in LUC(G)^*$. The following conditions are necessary for the left invertibility of $\mathfrak{m}$ in $LUC(G)^*$:*
(i) The operator $\mathfrak{m}_l \in \mathcal{B}(LUC(G))$ is bounded from below.
(ii) $j^*(\mathfrak{m}) \in Inv_l(M(G))$, where j denotes the inclusion map of $C_0(G)$ into $LUC(G)$.
(iii) $\mathfrak{m}_l^{-1}(C_0(G)) \subseteq C_0(G)$.

If G is an abelian locally compact group these conditions are also sufficient for the left invertibility of $\mathfrak{m}$ *in UC(G)*.*

Proof. Given $x \in$ LUC(G) and $h \in G$ we observe that

$$\begin{aligned} \| \mathfrak{m}_l(x) \| &= \sup_{g\in G} | \mathfrak{m}_l(x)(g) | \\ &= \sup_{g\in G} | \langle {}_g x, \mathfrak{m}\rangle | \\ &= \sup_{g\in G} | \langle {}_{hg} x, \mathfrak{m}\rangle | \\ &= \sup_{g\in G} | \langle {}_g({}_h x), \mathfrak{m}\rangle | \\ &= \| \mathfrak{m}_l({}_h x) \| . \end{aligned}$$

So the condition (i) of Theorem 2.1 holds if and only if there exists $c \in \mathbb{R}_{>0}$ such that $\| \mathfrak{m}_l(x) \| \geq c \| x \|$. Now the claim follows by Theorem 2.1. □

Corollary 2.3. *Let G be a locally compact abelian group and let* $\mathfrak{m} \in Inv_l(UC(G)^*)$*. Then* $ran(\mathfrak{m}_l)$ *is closed,* $C_0(G) \subseteq ran(\mathfrak{m}_l)$ *and*

$$\frac{ran(\mathfrak{m}_l)}{C_0(G)} \approx \frac{UC(G)}{C_0(G)}.$$

Proof. Since $\mathfrak{m}_l$ is bounded from below, its range is closed, and as seen in the proof of Theorem 2.1, it contains C_0(G). Let us define

$$\begin{aligned} &T_{\mathfrak{m}} : \mathrm{UC(G)}/\mathrm{C}_0(G) \to \mathrm{UC(G)}/\mathrm{C}_0(G), \\ &T_{\mathfrak{m}}(x + \mathrm{C}_0(G)) = \mathfrak{m}_l(x) + \mathrm{C}_0(G) \text{ if } x \in \mathrm{UC(G)}. \end{aligned}$$

Since $\mathrm{C}_0(G)$ is $\mathfrak{m}_l$-invariant $T_{\mathfrak{m}}$ is a well defined mapping that it is clearly complex linear. Furthermore, given $x \in$ UC(G) and $z \in \mathrm{C}_0(G)$ we see that

$$\| \mathfrak{m}_l(x) + \mathrm{C}_0(G) \| = \| \mathfrak{m}_l(x - z) + \mathrm{C}_0(G) \| \leq \| \mathfrak{m}_l \| \| x - z \|,$$

or in other words, $\| \mathfrak{m}_l(x) + \mathrm{C}_0(G) \| \leq \| \mathfrak{m}_l \| \| x + \mathrm{C}_0(G) \|$. It then follows that $T_{\mathfrak{m}} \in \mathcal{B}(\mathrm{UC(G)}/\mathrm{C}_0(\mathrm{G}))$ and $\| T_{\mathfrak{m}} \| \leq \| \mathfrak{m}_l \|$.
The set $\mathrm{ran}(T_{\mathfrak{m}}) = \mathrm{ran}(\mathfrak{m}_l)/\mathrm{C}_0(\mathrm{G})$ is thus closed and $\mathrm{ran}(T_{\mathfrak{m}}) \approx \mathrm{UC(G)}/\mathrm{C}_0(G)$ because, by Corollary 2.2, $T_{\mathfrak{m}}$ is injective. Finally, it suffices to observe that $\mathrm{ran}(T_{\mathfrak{m}}) = (\mathrm{ran}(\mathfrak{m}_l) + C_0(G))/\mathrm{C}_0(G)$. □

Corollary 2.4. *If* $\mathfrak{m} \in J(LUC(G)^*)$ *then* $j^*(\mathfrak{m}) \in J(M(G))$.

Proof. By the Hahn-Banach theorem j^* maps $\mathrm{LUC(G)}^*$ onto M(G). Furthermore, it is multiplicative. For, let $\mathfrak{l}, \mathfrak{n} \in \mathrm{LUC(G)}^*$ and $z \in \mathrm{C}_0$(G). Given $g \in G$ plainly ${}_g z \in \mathrm{C}_0(G)$ and

$$\mathfrak{n}_l(z)(g) = \langle {}_g z, \mathfrak{n}\rangle = \langle {}_g z, j^*(\mathfrak{n})\rangle = \int_G z(gh) dj^*(\mathfrak{n})(h).$$

By [**8**], Lemma 19.5, $\mathfrak{n}_l(z) \in \mathrm{C}_0(G)$, so by Fubini's theorem we can write

$$\begin{aligned}\langle z, j^*(\mathfrak{l}) * j^*(\mathfrak{n})\rangle &= \int_G \int_G z(gh) dj^*(\mathfrak{l})(g) dj^*(\mathfrak{n})(h)\\ &= \int_G \int_G z(gh) dj^*(\mathfrak{n})(h) dj^*(\mathfrak{l})(g)\\ &= \int_G \mathfrak{n}_l(z)(g) dj^*(\mathfrak{l})(g)\\ &= \langle \mathfrak{n}_l(z), \mathfrak{l}\rangle\\ &= \langle z, j^*(\mathfrak{l}\Box\mathfrak{n})\rangle\end{aligned}$$

and as z is arbitrary the claim follows.
Since $\mathfrak{m} \in \mathrm{J(LUC(G))}^*$ we know that $\hat{\delta_e} + \mathrm{LUC(G)}^*\Box\mathfrak{m} \subseteq \mathrm{Inv}_l(\mathrm{LUC(G)}^*)$. Thus

$$\delta_e + \mathrm{M(G)} * j^*(\mathfrak{m}) \subseteq \mathrm{Inv}_l(\mathrm{M(G)})$$

and $j^*(\mathfrak{m}) \in \mathrm{J(MG)}$ (cf. [**2**], Prop. 24.16(iii); [**4**], Th. 1.5.2(iii)). □

Corollary 2.5. *If G is abelian then $J(UC(G)^*) \subseteq C_0(G)^{\perp}$.*

Proof. If G is abelian the measure algebra M(G) becomes semisimple and the conclusion follows from (1.3) and Corollary 2.4 (cf. [**12**], A.3.3, p. 329). □

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