NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 56 (2026), 51–70
https://doi.org/10.53733/824

COUNTING INTEGERS OF CERTAIN FORMS WITH APPLICATIONS TO TERNARY QUADRATIC FORMS

Zafer Selcuk Aygin and Kenneth S. Williams

(Received 18 September, 2025)

Abstract. For ℓ and r fixed positive integers with $r < \ell$, we determine the number of positive integers $n \leq N$ which are of the forms $\ell^a(\ell b+r)$, $\ell^{2a}(\ell b+r)$, $\ell^{2a+1}(\ell b+r)$ for some nonnegative integers a and b; and for fixed $r \in \{1, 3, 5, 7\}$ we find the number of positive integers $n \leq N$ of each of the forms $2^{2a}(8b+r)$, and $2^{2a+1}(8b+r)$ for some nonnegative integers a and b. Then we use our results to count the number of positive integers $n \leq N$ which can be represented by certain ternary quadratic forms.

1. Introduction

As is customary, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the sets of integers, positive integers, and nonnegative integers, respectively. Throughout this paper N and ℓ are integers satisfying $N \geq 1,\ \ell \geq 2$. The representation of N in base ℓ is

$$N = a_0 + a_1\ell + a_2\ell^2 + \cdots + a_k\ell^k = (a_0a_1 \ldots a_k), \tag{1.1}$$

where $k \in \mathbb{N}_0$, each $a_i \in \{0, 1, \ldots, \ell-1\}$, and $a_k \neq 0$. We define for $j \in \{0, 1, \ldots, \ell-1\}$

$$\mathcal{A}_{N,\ell}(j) := \sum_{\substack{i=0 \\ a_i=j}}^{k} 1, \quad \mathcal{B}_{N,\ell}(j) := \sum_{\substack{i=0 \\ a_{2i}=j}}^{[k/2]} 1, \quad \mathcal{C}_{N,\ell}(j) := \sum_{\substack{i=0 \\ a_{2i+1}=j}}^{[(k-1)/2]} 1, \tag{1.2}$$

where for a real number x, $[x]$ denotes the largest integer less than or equal to x. In this notation, when ℓ is clear from the context, we sometimes drop it as the subscript. We note that $\mathcal{B}_{N,\ell}(j)$ and $\mathcal{C}_{N,\ell}(j)$ are related to $\mathcal{A}_{N,\ell}(j)$ by

$$\mathcal{A}_{N,\ell}(j) = \mathcal{B}_{N,\ell}(j) + \mathcal{C}_{N,\ell}(j). \tag{1.3}$$

The number of positive integers $n \leq N$ having a particular form $F(a, b)$ for some nonnegative integers a and b is denoted by $ct_N(F(a, b))$, that is,

$$ct_N(F(a, b)) = \#\{n \in \mathbb{N} : n \leq N \text{ and } n = F(a, b) \text{ for some } a, b \in \mathbb{N}_0\}.$$

It is a classical theorem of Legendre that only positive integers of the form $2^{2a}(8b+7)$ $(a, b \in \mathbb{N}_0)$ cannot be represented as a sum of three squares. Thus, the number of $n \in \mathbb{N}$ with $n \leq N$ which are the sum of three squares is

$$N - ct_N(2^{2a}(8b + 7)).$$

2020 *Mathematics Subject Classification* 11A67, 11D85, 11E25.

Key words and phrases: Representation of integers.
The research of Z.S. Aygin is supported by the American University of Sharjah, FSU Grant No: SC2508. The authors thank the referees for their helpful comments, which led to improvements in the presentation of this article.

In 1908 Landau [**4**] proved that

$$\lim_{N\to\infty} \frac{ct_N(2^{2a}(8b+7))}{N} = \frac{1}{6}.$$

In 1940, M.C. Chakrabarti [**1**, eq. (2), p. 2] proved as a consequence of his work on sums of three squares that

$$ct_N(2^{2a}(8b+7)) = \frac{1}{6}N - \frac{2}{3}\mathcal{B}_N(1) - \frac{1}{3}\mathcal{C}_N(1) + \mathcal{E}_N(111) + \frac{1}{2}a_0,$$

where for $u, v, w \in \{0, 1\}$

$$\mathcal{E}_N(uvw) := \sum_{a_{2r-2}a_{2r-1}a_{2r}=uvw} 1 \quad \text{and} \quad \mathcal{O}_N(uvw) := \sum_{a_{2r-1}a_{2r}a_{2r+1}=uvw} 1.$$

As noted by Shiu [**5**, p. 203], Chakrabarti also proved the inequalities

$$-\frac{\log(N)}{3\log(2)} - \frac{1}{2} < ct_N(2^{2a}(8b+7)) - \frac{N}{6} \le -\frac{1}{6}.$$

Grosswald [**3**, Ch. 4, Theorem 6] gave the formula

$$ct_N(2^{2a}(8b+7)) = \frac{1}{6}N - \frac{7}{8\log(4)}\log(N) + \frac{7\log(7)}{8\log(4)} - \frac{7}{6} + R(N),$$

where

$$0 \le R(N) \le \frac{\log(N/7)}{\log(4)} + 1.$$

In this paper, we extend these results to three infinite families of forms and two families of forms related to the result discussed above. Namely, we determine formulas for $ct_N(F)$ when

$$F(a,b) \in \{F_1(a,b), F_2(a,b), F_3(a,b), F_4(a,b), F_5(a,b)\},$$

where

$$F_1(a,b) := \ell^a(\ell b + r), \tag{1.4}$$
$$F_2(a,b) := \ell^{2a}(\ell b + r), \tag{1.5}$$
$$F_3(a,b) := \ell^{2a+1}(\ell b + r), \tag{1.6}$$

for $r \in \{1, \dots, \ell - 1\}$, and

$$F_4(a,b) := 2^{2a}(8b + r), \tag{1.7}$$
$$F_5(a,b) := 2^{2a+1}(8b + r), \tag{1.8}$$

for $r \in \{1, 3, 5, 7\}$.

With the form as in (1.4) the explicit evaluation of $ct_N(F_1)$ can be given in terms of $\mathcal{A}_N(j)$ $(j \in \{0, 1, \dots, \ell - 1\})$. In Section 3, we prove the following result.

Theorem 1.1. *For $r \in \{1, 2, \dots, \ell - 1\}$, we have*

$$ct_N(\ell^a(\ell b + r)) = \frac{N}{\ell - 1} + \sum_{j=r}^{\ell-1} \mathcal{A}_N(j) - \sum_{j=1}^{\ell-1} \frac{j}{\ell - 1}\mathcal{A}_N(j). \tag{1.9}$$

For the forms as in (1.5) and (1.6) the evaluation of $ct_N(F_2)$ and $ct_N(F_3)$ requires the quantities $\mathcal{B}_N(j)$ and $\mathcal{C}_N(j)$ ($j \in \{0, 1, \ldots, \ell-1\}$). In Section 3, we prove the following theorem.

Theorem 1.2. *For $r \in \{1, 2, \ldots, \ell-1\}$, we have*

$$ct_N(\ell^{2a}(\ell b + r)) = \frac{\ell N}{\ell^2 - 1} + \sum_{j=r}^{\ell-1} \mathcal{B}_N(j) - \sum_{j=1}^{\ell-1} \left(\frac{j\ell}{\ell^2-1} \mathcal{B}_N(j) + \frac{j}{\ell^2-1} \mathcal{C}_N(j) \right), \tag{1.10}$$

and

$$ct_N(\ell^{2a+1}(\ell b + r)) = \frac{N}{\ell^2 - 1} + \sum_{j=r}^{\ell-1} \mathcal{C}_N(j) - \sum_{j=1}^{\ell-1} \left(\frac{j}{\ell^2-1} \mathcal{B}_N(j) + \frac{j\ell}{\ell^2-1} \mathcal{C}_N(j) \right). \tag{1.11}$$

For the forms as in (1.7) and (1.8) we require the quantities $\mathcal{E}_N(uvw)$ and $\mathcal{O}_N(uvw)$ in order to determine $ct_N(F_4)$ and $ct_N(F_5)$. We clarify that when counting the quantities $\mathcal{E}_N(uvw)$ and $\mathcal{O}_N(uvw)$, we consider $a_{k+1} = 0$ and $a_{k+2} = 0$ in the sequence corresponding to N's representation in base 2. For example, for

$$N = 5 = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 = (101) = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 = (10100),$$

we have $\mathcal{E}_5(100) = 1$.

Theorem 1.3. *For $r \in \{1, 3, 5, 7\}$, we have*

$$\begin{aligned} ct_N(2^{2a}(8b + r)) = \frac{1}{6}N + \frac{a_1}{6}\mathcal{E}_N(100) + \frac{a_2}{3}\mathcal{E}_N(010) + \frac{a_3}{2}\mathcal{E}_N(001) + \frac{a_4}{2}\mathcal{E}_N(110) \\ + \frac{a_5}{3}\mathcal{E}_N(101) + \frac{a_6}{6}\mathcal{E}_N(011), \end{aligned}$$

and

$$\begin{aligned} ct_N(2^{2a+1}(8b + r)) = \frac{1}{12}N + \frac{a_1}{6}\mathcal{O}_N(100) + \frac{a_2}{3}\mathcal{O}_N(010) + \frac{a_3}{2}\mathcal{O}_N(001) + \frac{a_4}{2}\mathcal{O}_N(110) \\ + \frac{a_5}{3}\mathcal{O}_N(101) + \frac{a_6}{6}\mathcal{O}_N(011) - \frac{1-(-1)^N}{24}, \end{aligned}$$

where the values of a_1, a_2, a_3, a_4, a_5, and a_6 for each $r \in \{1, 3, 5, 7\}$ are given in Table 1.

r	a_1	a_2	a_3	a_4	a_5	a_6
1	5	2	1	1	1	1
3	−1	−1	1	1	1	1
5	−1	−1	−1	−1	1	1
7	−1	−1	−1	−1	−2	−5

TABLE 1. Values of $a_1, a_2, a_3, a_4, a_5, a_6$ for each r.

Theorem 1.3 is proved in Section 4. Equating the formula for $ct_N(2^{2a}(8b+7))$ in Theorem 1.3 to Chakrabarti's formula, we obtain, through the easily proved identities

$$\mathcal{B}_N(1) = \mathcal{E}_N(100) + \mathcal{E}_N(101) + \mathcal{E}_N(110) + \mathcal{E}_N(111)$$

and

$$\mathcal{C}_N(1) = \mathcal{E}_N(010) + \mathcal{E}_N(011) + \mathcal{E}_N(110) + \mathcal{E}_N(111),$$

the following somewhat surprising result.

Corollary 1.1. *If* $N = a_0 + a_1 2 + \cdots + a_k 2^k$ *then*

$$a_0 = \mathcal{E}_N(100) - \mathcal{E}_N(001) + \mathcal{E}_N(110) - \mathcal{E}_N(011).$$

Next, we discuss the asymptotic behavior of $ct_N(F)$. We have $N \geq \ell^k$ so $\log_\ell(N) \geq k$ and thus

$$0 \leq \mathcal{A}_{N,\ell}(j) \leq k+1 \leq \log_\ell(\ell N).$$

Similarly we have

$$0 \leq \mathcal{B}_{N,\ell}(j),\ \mathcal{C}_{N,\ell}(j) \leq \log_\ell(\ell^2 N)/2,$$

and

$$0 \leq \mathcal{E}_N(uvw), \mathcal{O}_N(uvw) \leq \log_2(4N)/2.$$

As $\log_\ell(N)$ is much smaller than N for large N, our formulas are useful in estimating the size of $ct_N(F)$.

Corollary 1.2. *We have*

$$-\frac{r(r-1)}{2(\ell-1)}\log_\ell(\ell N) \leq ct_N(\ell^a(\ell b+r)) - \frac{N}{\ell-1} \leq \frac{(\ell-r-1)(\ell-r)}{2(\ell-1)}\log_\ell(\ell N),$$

$$-\frac{\ell(r^2-r+\ell-1)}{4(\ell^2-1)}\log_\ell\left(\ell^2 N\right) \leq ct_N(\ell^{2a}(\ell b+r)) - \frac{\ell N}{\ell^2-1} \leq (\ell-r)\left(\frac{\ell^2-\ell(r-1)-2}{4(\ell^2-1)}\right)\log_\ell(\ell^2 N),$$

$$-\frac{\ell(r^2-r+\ell-1)}{4(\ell^2-1)}\log_\ell\left(\ell^2 N\right) \leq ct_N(\ell^{2a+1}(\ell b+r)) - \frac{N}{\ell^2-1} \leq (\ell-r)\left(\frac{\ell^2-\ell(r-1)-2}{4(\ell^2-1)}\right)\log_\ell(\ell^2 N),$$

$$0 \leq ct_N(2^{2a}(8b+1)) - \frac{N}{6} \leq \frac{3}{2}\log_2(4N),$$

$$-\frac{1}{4}\log_2(4N) \leq ct_N(2^{2a}(8b+3)) - \frac{N}{6} \leq \frac{3}{4}\log_2(4N),$$

$$-\frac{3}{4}\log_2(4N) \leq ct_N(2^{2a}(8b+5)) - \frac{N}{6} \leq \frac{1}{4}\log_2(4N),$$

$$-\frac{3}{2}\log_2(4N) \leq ct_N(2^{2a}(8b+7)) - \frac{N}{6} \leq 0,$$

$$-\frac{1}{12} \leq ct_N(2^{2a+1}(8b+1)) - \frac{N}{12} \leq \frac{3}{2}\log_2(4N),$$

$$-\frac{1}{4}\log_2(4N) - \frac{1}{12} \leq ct_N(2^{2a+1}(8b+3)) - \frac{N}{12} \leq \frac{3}{4}\log_2(4N),$$

$$-\frac{3}{4}\log_2(4N) - \frac{1}{12} \leq ct_N(2^{2a+1}(8b+5)) - \frac{N}{12} \leq \frac{1}{4}\log_2(4N),$$

$$-\frac{3}{2}\log_2(4N) - \frac{1}{12} \leq ct_N(2^{2a+1}(8b+7)) - \frac{N}{12} \leq 0.$$

We note that the size of the error in both Grosswald's estimate and our estimate for $ct_N(2^{2a}(8b+7)) - N/6$ is the same, namely $O(\log(N))$.

As $\lim_{N\to\infty}\log_\ell(N)/N = 0$, Corollary 1.2 gives the following result.

Corollary 1.3. *For* $r \in \{1, 2, \ldots, \ell-1\}$ *we have*

$$\lim_{N\to\infty}\frac{ct_N(\ell^a(\ell b+r))}{N} = \frac{1}{\ell-1},$$

$$\lim_{N\to\infty}\frac{ct_N(\ell^{2a}(\ell b+r))}{N} = \frac{\ell}{\ell^2-1},$$

and

$$\lim_{N\to\infty} \frac{ct_N(\ell^{2a+1}(\ell b+r))}{N} = \frac{1}{\ell^2-1}.$$

For $r \in \{1,3,5,7\}$ *we have*

$$\lim_{N\to\infty} \frac{ct_N(2^{2a}(8b+r))}{N} = \frac{1}{6},$$

and

$$\lim_{N\to\infty} \frac{ct_N(2^{2a+1}(8b+r))}{N} = \frac{1}{12}.$$

2. Applications of Theorems 1.2 and 1.3 to Counting the Number of Positive Integers $n \le N$ which are Represented by some Ternary Quadratic Forms

In this section, we apply Theorems 1.2 and 1.3 to the problem of counting the number of positive integers $n \le N$ which are represented by certain positive-definite primitive integral ternary quadratic forms

$$t(x,y,z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy. \tag{2.1}$$

We denote this count by

$$ct_N(a,b,c,d,e,f) := \#\{n \le N : t(x,y,z) = n \text{ for some } x,y,z \in \mathbb{Z}\}.$$

Tables 1 and 2 in [**2**, Appendix] contain 63 ternary quadratic forms (2.1) to which Theorems 1.2 and 1.3 can be applied.

Theorem 2.1. *(i) For the* 11 *ternary quadratic forms* $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ *listed in Table 2, we have*

$$ct_N(a,b,c,d,e,f) = \frac{5}{6}N + \frac{a_1}{6}\mathcal{E}_N(100) + \frac{a_2}{3}\mathcal{E}_N(010) + \frac{a_3}{2}\mathcal{E}_N(001) + \frac{a_4}{2}\mathcal{E}_N(110) + \frac{a_5}{3}\mathcal{E}_N(101) + \frac{a_6}{6}\mathcal{E}_N(011),$$

where the values a_1,a_2,a_3,a_4,a_5,a_6 *are given in the same table.*

a	b	c	d	e	f	a_1	a_2	a_3	a_4	a_5	a_6
1	1	1	0	0	0	1	1	1	1	2	5
1	1	5	0	0	0	1	1	−1	−1	−1	−1
1	2	2	0	0	0	1	1	1	1	2	5
1	2	2	2	0	0	1	1	1	1	−1	−1
1	2	5	2	0	0	1	1	1	1	2	5
1	2	6	0	0	0	1	1	1	1	−1	−1
1	2	6	2	0	0	1	1	1	1	−1	−1
2	2	3	2	2	2	−5	−2	−1	−1	−1	−1
2	3	3	0	0	2	−5	−2	−1	−1	−1	−1
2	3	5	2	0	0	−5	−2	−1	−1	−1	−1
2	3	5	2	0	2	−5	−2	−1	−1	−1	−1

TABLE 2. Values of a,b,c,d,e,f and a_1,a_2,a_3,a_4,a_5,a_6.

(ii) *For the 12 ternary quadratic forms* $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ *listed in Table 3, we have*

$$ct_N(a,b,c,d,e,f) = \frac{11}{12}N + \frac{b_1}{6}\mathcal{O}_N(100) + \frac{b_2}{3}\mathcal{O}_N(010) + \frac{b_3}{2}\mathcal{O}_N(001) + \frac{b_4}{2}\mathcal{O}_N(110) + \frac{b_5}{3}\mathcal{O}_N(101) + \frac{b_6}{6}\mathcal{O}_N(011) + \frac{1-(-1)^N}{24},$$

where the values $b_1, b_2, b_3, b_4, b_5, b_6$ *are given in the same table.*

a	b	c	d	e	f	b_1	b_2	b_3	b_4	b_5	b_6
1	1	1	1	1	1	1	1	1	1	2	5
1	1	2	0	0	0	1	1	1	1	2	5
1	1	2	0	0	1	1	1	1	1	-1	-1
1	1	3	1	1	0	1	1	-1	-1	-1	-1
1	1	5	1	1	1	-5	-2	-1	-1	-1	-1
1	2	3	0	0	0	1	1	1	1	-1	-1
1	2	3	0	1	0	1	1	1	1	-1	-1
1	2	3	2	1	0	1	1	1	1	2	5
1	2	4	0	0	0	1	1	1	1	2	5
1	3	3	1	1	1	-5	-2	-1	-1	-1	-1
1	3	5	2	0	0	-5	-2	-1	-1	-1	-1
1	3	5	3	1	1	-5	-2	-1	-1	-1	-1

TABLE 3. Values of a, b, c, d, e, f and $b_1, b_2, b_3, b_4, b_5, b_6$.

(iii) *For the 20 ternary quadratic forms* $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ *listed in Table 4, we have*

$$ct_N(a,b,c,d,e,f) = \frac{(\ell+2)N}{2(\ell+1)} + \sum_{j=1}^{\ell-1} \frac{j\left(\ell\mathcal{B}_N(j) + \mathcal{C}_N(j)\right)}{2(\ell+1)} - \sum_{r=1}^{\ell-1}\sum_{j=r}^{\ell-1} \alpha(r)\mathcal{B}_N(j)$$

where the values a, b, c, d, e, f *and* $\ell, \alpha(r)$ *are given in the same table and* $\chi(r)$ *denotes the Legendre symbol* $\left(\frac{r}{\ell}\right)$.

a	b	c	d	e	f	ℓ	$\alpha(r)$	a	b	c	d	e	f	ℓ	$\alpha(r)$
2	2	2	-1	1	1	5	$(1+\chi(r))/2$	1	1	3	0	0	1	3	$(1-\chi(r))/2$
2	2	2	1	2	2	3	$(1+\chi(r))/2$	1	2	7	0	0	1	7	$(1-\chi(r))/2$
2	2	3	0	0	1	3	$(1+\chi(r))/2$	1	3	3	0	0	0	3	$(1-\chi(r))/2$
2	2	5	1	1	1	3	$(1+\chi(r))/2$	1	3	4	3	1	0	3	$(1-\chi(r))/2$
2	2	7	-1	1	1	5	$(1+\chi(r))/2$	1	3	6	3	0	0	3	$(1-\chi(r))/2$
2	3	3	0	0	0	3	$(1+\chi(r))/2$	1	3	11	0	0	1	11	$(1-\chi(r))/2$
2	3	5	0	0	2	5	$(1+\chi(r))/2$	1	4	4	3	1	1	5	$(1-\chi(r))/2$
2	3	5	3	1	0	3	$(1+\chi(r))/2$	1	4	5	0	0	1	5	$(1-\chi(r))/2$
2	5	5	-3	1	1	13	$(1+\chi(r))/2$	1	5	10	0	0	0	5	$(1-\chi(r))/2$
3	3	5	-2	2	1	7	$(1+\chi(r))/2$								
3	5	6	1	2	3	17	$(1+\chi(r))/2$								

TABLE 4. Values of a, b, c, d, e, f and $\ell, \alpha(r)$.

(iv) *For the 20 ternary quadratic forms* $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ *listed in Table 5, we have*

$$ct_N(a,b,c,d,e,f) = \frac{(2\ell+1)N}{2(\ell+1)} + \sum_{j=1}^{\ell-1} \frac{j\left(\mathcal{B}_N(j) + \ell\mathcal{C}_N(j)\right)}{2(\ell+1)} - \sum_{r=1}^{\ell-1}\sum_{j=r}^{\ell-1} \alpha(r)\mathcal{C}_N(j)$$

where the values a, b, c, d, e, f and $\ell, \alpha(r)$ are given in the same table and $\chi(r)$ denotes the Legendre symbol $\left(\frac{r}{\ell}\right)$.

a	b	c	d	e	f	ℓ	$\alpha(r)$	a	b	c	d	e	f	ℓ	$\alpha(r)$
1	1	2	1	1	0	3	$(1+\chi(r))/2$	1	1	1	0	0	1	3	$(1-\chi(r))/2$
1	1	2	1	1	1	5	$(1+\chi(r))/2$	1	1	2	0	1	0	7	$(1-\chi(r))/2$
1	1	4	0	1	0	3	$(1+\chi(r))/2$	1	1	3	0	0	0	3	$(1-\chi(r))/2$
1	1	6	0	0	0	3	$(1+\chi(r))/2$	1	1	3	0	1	0	11	$(1-\chi(r))/2$
1	2	2	1	0	1	13	$(1+\chi(r))/2$	1	2	2	1	0	0	5	$(1-\chi(r))/2$
1	2	3	1	0	1	5	$(1+\chi(r))/2$	1	2	2	1	1	1	3	$(1-\chi(r))/2$
1	2	3	1	1	0	7	$(1+\chi(r))/2$	1	2	2	2	1	1	5	$(1-\chi(r))/2$
1	2	3	2	0	0	5	$(1+\chi(r))/2$	1	2	3	0	0	1	3	$(1-\chi(r))/2$
1	2	3	2	1	1	17	$(1+\chi(r))/2$	1	2	5	0	0	0	5	$(1-\chi(r))/2$
1	2	4	2	1	1	3	$(1+\chi(r))/2$								
1	2	5	1	1	1	3	$(1+\chi(r))/2$								

TABLE 5. Values of a, b, c, d, e, f and $\ell, \alpha(r)$.

Proof. As the proofs are similar for all 63 ternary quadratic forms, we just give the proof for $(a, b, c, d, e, f) = (2, 5, 5, -3, 1, 1)$. By [**2**, Table 1, Entry 772], we have

$$ct_N(2, 5, 5, -3, 1, 1) = N - \sum_{r \in \{1,3,4,9,10,12\}} ct_N(13^{2a}(13b + r)).$$

As all the r's in the summation are quadratic residues modulo 13, we can write this as

$$ct_N(2, 5, 5, -3, 1, 1) = N - \sum_{r=1}^{\ell-1} \frac{1 + \chi(r)}{2} ct_N(13^{2a}(13b + r)).$$

Then we appeal to Theorem 1.2 to obtain the desired result. $\square$

3. Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2 using mathematical induction. We now explain the general idea of how the proofs work. We denote the formulas given in Theorems 1.1 and 1.2 by $W(N)$. We wish to prove that $W(N) = ct_N(F)$ for all $N \in \mathbb{N}$. The base cases $W(1) = ct_1(F)$ are almost trivial. For the inductive step, we assume $ct_N(F) = W(N)$ for some $N \geq 1$. Then, we observe that

$$ct_{N+1}(F) - ct_N(F) = \begin{cases} 0 & \text{if } N+1 \text{ is not of the form } F, \\ 1 & \text{if } N+1 \text{ is of the form } F, \end{cases}$$

and, using properties of $\mathcal{A}_N(j)$, $\mathcal{B}_N(j)$ and $\mathcal{C}_N(j)$, we show that

$$W(N+1) - W(N) = \begin{cases} 0 & \text{if } N+1 \text{ is not of the form } F, \\ 1 & \text{if } N+1 \text{ is of the form } F. \end{cases}$$

By the inductive hypothesis, we deduce that $W(N + 1) = ct_{N+1}(F)$, completing the proof.

Proof of Theorem 1.1. We begin by defining

$$W(N) := \frac{N}{\ell-1} + \sum_{j=r}^{\ell-1} \mathcal{A}_N(j) - \frac{1}{\ell-1}\sum_{j=1}^{\ell-1} j\mathcal{A}_N(j), \tag{3.1}$$

so that

$$W(N+1)-W(N) = \frac{1}{\ell-1} + \sum_{j=r}^{\ell-1}(\mathcal{A}_{N+1}(j)-\mathcal{A}_N(j)) - \frac{1}{\ell-1}\sum_{j=1}^{\ell-1} j(\mathcal{A}_{N+1}(j)-\mathcal{A}_N(j)).$$

By induction on N we prove $ct_N(F_1) = W(N),$ for all $N \in \mathbb{N}$.

For $N=1$ we have $a_0 = 1$ and $k=0$ so that

$$\mathcal{A}_N(j) = \begin{cases} 1 & \text{if } j=1, \\ 0 & \text{if } j \neq 1. \end{cases}$$

Thus

$$W(1) = \begin{cases} 1 & \text{if } r=1, \\ 0 & \text{if } r \neq 1. \end{cases}$$

Now as

$$1 = \ell^a(\ell b + r) \text{ if and only if } a=0,\ b=0,\ r=1,$$

we have

$$ct_1(F_1) = \begin{cases} 1 & \text{if } r=1, \\ 0 & \text{if } r \neq 1. \end{cases}$$

Hence

$$ct_1(F_1) = W(1),$$

which completes the proof of the starting point of the induction.

The inductive hypothesis is

$$ct_N(F_1) = W(N) \text{ for some } N \geq 1.$$

We wish to prove that $ct_{N+1}(F_1) = W(N+1)$.

We consider four cases:

- Case 1: $a_0 = 0$.
- Case 2: $a_0 = 1, 2, \ldots, \ell-2$.
- Case 3: $a_0 = a_1 = a_2 = \cdots = a_k = \ell-1$.
- Case 4: $a_0 = a_1 = a_2 = \cdots = a_s = \ell-1,\ a_{s+1} \neq \ell-1$ for some s with $0 \leq s \leq k-1$.

We prove that $ct_{N+1}(F_1) - ct_N(F_1) = W(N+1) - W(N)$ in each case.

Case 1: If $a_0 = 0$, then by (1.1) we have

$$N+1 = 1 + a_1\ell + a_2\ell^2 + \cdots + a_k\ell^k$$

so that $N+1 \equiv 1 \pmod{\ell}$. Hence if $r \neq 1$ then $N+1 \neq \ell^a(b\ell+r)$ for any $a, b \in \mathbb{N}_0$, whereas if $r=1$ then $N+1 = \ell^a(b\ell+r)$ with $a=0$ and $b = N/\ell$, proving that

$$ct_{N+1}(F_1) - ct_N(F_1) = \begin{cases} 1 & \text{if } r=1, \\ 0 & \text{if } r \neq 1. \end{cases} \tag{3.2}$$

Now by definition of $A_N(j)$ we see that

$$\mathcal{A}_{N+1}(j) = \begin{cases} \mathcal{A}_N(j)+1 & \text{if } j=1, \\ \mathcal{A}_N(j) & \text{if } j=2,\dots,\ell-1, \end{cases}$$

so

$$W(N+1)-W(N) = \begin{cases} 1 & \text{if } r=1, \\ 0 & \text{if } r\neq 1. \end{cases} \tag{3.3}$$

Hence, by (3.2) and (3.3) we have

$$ct_{N+1}(F_1)-ct_N(F_1) = W(N+1)-W(N).$$

Case 2: If $a_0 = 1,2,\dots,\ell-2$, then as $a_0+1<\ell$ by (1.1) we have

$$N+1 = (a_0+1)+a_1\ell+a_2\ell^2+\cdots+a_k\ell^k,$$

so that $N+1\equiv a_0+1 \pmod{\ell}$. In a similar manner to the proof of Case 1, we obtain

$$N+1 \begin{cases} \neq \ell^a(b\ell+r) & \text{if } r\neq a_0+1, \\ = \ell^a(b\ell+r) & \text{with } a=b=0 \text{ if } r=a_0+1, \end{cases}$$

$$ct_{N+1}(F_1)-ct_N(F_1) = \begin{cases} 0 & \text{if } r\neq a_0+1, \\ 1 & \text{if } r=a_0+1, \end{cases} \tag{3.4}$$

$$\mathcal{A}_{N+1}(j) = \begin{cases} \mathcal{A}_N(j)-1 & \text{if } j=a_0, \\ \mathcal{A}_N(j) & \text{if } j\neq a_0, a_0+1, \\ \mathcal{A}_N(j)+1 & \text{if } j=a_0+1, \end{cases}$$

$$W(N+1)-W(N) = \begin{cases} 0 & \text{if } r\neq a_0+1, \\ 1 & \text{if } r=a_0+1. \end{cases} \tag{3.5}$$

Hence, by (3.4) and (3.5) we have

$$ct_{N+1}(F_1)-ct_N(F_1) = W(N+1)-W(N).$$

Case 3: If $a_0=a_1=a_2=\ldots=a_k=\ell-1$, then

$$N=(\ell-1)+(\ell-1)\ell+(\ell-1)\ell^2+\cdots+(\ell-1)\ell^k, \quad N+1=\ell^{k+1}.$$

Similarly to Cases 1 and 2, we see that $N+1=\ell^a(b\ell+r)$, if and only if $a=k+1$, $b=0$, and $r=1$. Thus

$$ct_{N+1}(F_1)-ct_N(F_1) = \begin{cases} 0 & \text{if } r\neq 1, \\ 1 & \text{if } r=1. \end{cases} \tag{3.6}$$

Also

$$\mathcal{A}_N(j) = \begin{cases} k+1 & \text{if } j=\ell-1, \\ 0 & \text{if } j\neq \ell-1, \end{cases} \quad \mathcal{A}_{N+1}(j) = \begin{cases} 1 & \text{if } j=1, \\ 0 & \text{if } j\neq 1, \end{cases}$$

so for $\ell \neq 2$ (note $\ell - 1 \neq 1$)

$$\mathcal{A}_{N+1}(j) - \mathcal{A}_N(j) = \begin{cases} 1 & \text{if } j = 1, \\ -k-1 & \text{if } j = \ell - 1, \\ 0 & \text{if } j \neq 1, \ell - 1, \end{cases}$$

and for $\ell = 2$, j can only be 1, so

$$\mathcal{A}_{N+1}(j) - \mathcal{A}_N(j) = -k.$$

Thus, both when $\ell \neq 2$ and $\ell = 2$, we have

$$W(N+1) - W(N) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \neq 1. \end{cases} \tag{3.7}$$

Hence, by (3.6) and (3.7) we have

$$ct_{N+1}(F_1) - ct_N(F_1) = W(N+1) - W(N).$$

Case 4: $a_0 = a_1 = a_2 = \ldots = a_s = \ell - 1$, $a_{s+1} \neq \ell - 1$ for some s with $0 \leq s \leq k-1$.
In this case, we have

$$N = (\ell - 1) + (\ell - 1)\ell + (\ell - 1)\ell^2 + \cdots + (\ell - 1)\ell^s + a_{s+1}\ell^{s+1} + \cdots + a_k\ell^k,$$

and

$$N + 1 = (a_{s+1} + 1)\ell^{s+1} + \cdots + a_k\ell^k,$$

where $a_{s+1} + 1 < \ell$ and $a_k \neq 0$. We have $N + 1 = \ell^a(\ell b + r)$ for some $a, b \in \mathbb{N}_0$, if and only if $s + 1 = a$, $b = 0$, and $r = a_{s+1} + 1$ so that

$$ct_{N+1}(F_1) - ct_N(F_1) = \begin{cases} 1 & \text{if } r = a_{s+1} + 1, \\ 0 & \text{if } r \neq a_{s+1} + 1. \end{cases} \tag{3.8}$$

Next we let $M := a_{s+2}\ell^{s+2} + \cdots + a_k\ell^k$ so that

$$N = (\ell - 1) + (\ell - 1)\ell + (\ell - 1)\ell^2 + \cdots + (\ell - 1)\ell^s + a_{s+1}\ell^{s+1} + M,$$

and

$$N + 1 = (a_{s+1} + 1)\ell^{s+1} + M,$$

where $a_{s+1} < \ell - 1$. Hence we have for $j \in \{1, 2, \ldots, \ell - 1\}$

$$\mathcal{A}_N(j) = \begin{cases} \mathcal{A}_M(j) + 1 \text{ if } j = a_{s+1}, \\ \mathcal{A}_M(j) + s + 1 \text{ if } j = \ell - 1, \\ \mathcal{A}_M(j) \text{ if } j \neq a_{s+1}, \ell - 1, \end{cases}$$

so that for $a_{s+1} \neq \ell - 2$ (that is, $a_{s+1} + 1 \neq \ell - 1$)

$$\mathcal{A}_{N+1}(j) - \mathcal{A}_N(j) = \begin{cases} -1 & \text{if } j = a_{s+1}, \\ 1 & \text{if } j = a_{s+1} + 1, \\ -s-1 & \text{if } j = \ell - 1, \\ 0 & \text{if } j \neq a_{s+1}, a_{s+1} + 1, \ell - 1, \end{cases}$$

whereas if $a_{s+1} = \ell - 2$ (so that $a_{s+1} + 1 = \ell - 1$) we have

$$\mathcal{A}_{N+1}(j) - \mathcal{A}_N(j) = \begin{cases} -1 & \text{if } j = a_{s+1}, \\ -s & \text{if } j = a_{s+1} + 1, \ell - 1, \\ 0 & \text{if } j \neq a_{s+1}, a_{s+1} + 1, \ell - 1. \end{cases}$$

Thus, using these in (3.1), for $a_{s+1} \neq \ell - 2$ we have

$$W(N+1) - W(N) = \begin{cases} 1 & \text{if } r = a_{s+1} + 1, \\ 0 & \text{if } r \neq a_{s+1} + 1, \end{cases} \tag{3.9}$$

and for $a_{s+1} = \ell - 2$ we have

$$W(N+1) - W(N) = \begin{cases} 1 & \text{if } r = \ell - 1, \\ 0 & \text{if } r \neq \ell - 1. \end{cases} \tag{3.10}$$

Hence, by (3.8), (3.9) and (3.10) we have

$$ct_{N+1}(F_1) - ct_N(F_1) = W(N+1) - W(N).$$

Therefore, in all four cases, we have

$$ct_{N+1}(F_1) - ct_N(F_1) = W(N+1) - W(N).$$

By the inductive hypothesis this means $ct_{N+1}(F_1) = W(N+1)$. Hence, by mathematical induction, for all $N \geq 1$ we have

$$ct_N(F_1) = \frac{N}{\ell - 1} + \sum_{j=r}^{\ell-1} \mathcal{A}_N(j) - \frac{1}{\ell - 1} \sum_{j=1}^{\ell-1} j\mathcal{A}_N(j),$$

which is Theorem 1.1. □

Proof of Theorem 1.2. We begin by defining

$$W(N) := \frac{\ell}{\ell^2 - 1} N + \sum_{j=r}^{\ell-1} \mathcal{B}_N(j) - \sum_{j=1}^{\ell-1} \left(\frac{j\ell}{\ell^2 - 1} \mathcal{B}_N(j) + \frac{j}{\ell^2 - 1} \mathcal{C}_N(j) \right). \tag{3.11}$$

We are going to prove that $ct_N(F_2) = W(N)$ for all $N \in \mathbb{N}$ by induction on N.

For $N = 1$ we have $a_0 = 1$ and $k = 0$, so that by (1.1) and (1.2) we have

$$\mathcal{B}_1(j) = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j \neq 1, \end{cases} \quad \mathcal{C}_1(j) = 0.$$

Thus

$$W(1) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \neq 1. \end{cases}$$

From the definition, we have

$$ct_1(F_2) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \neq 1, \end{cases}$$

so $ct_1(F_2) = W(1)$ proving the base case for the induction. The inductive hypothesis is

$$ct_N(F_2) = W(N) \text{ for some } N \geq 1. \tag{3.12}$$

We wish to prove that $ct_{N+1}(F_2) = W(N+1)$.

We consider four cases:

- Case 1: $a_0 = 0$. (As $a_k \neq 0$ we have $k \geq 1$.)
- Case 2: $a_0 = 1, 2, \ldots, \ell - 2$. (As $a_0 \neq 0$ we have $\ell \geq 3$.)
- Case 3: $a_0 = \ell - 1$, $a_1 = a_2 = \ldots = a_k = \ell - 1$.
- Case 4: $a_0 = \ell - 1$, $a_1 = a_2 = \ldots = a_s = \ell - 1$, $a_{s+1} \neq \ell - 1$ for some s with $0 \leq s \leq k - 1$.

We prove that $W(N+1) - W(N) = ct_{N+1}(F_2) - ct_N(F_2)$ in each case.

Case 1: $a_0 = 0$.

In this case, we have

$$N = 0 + a_1\ell + a_2\ell^2 + \cdots + a_k\ell^k, \quad N + 1 = 1 + a_1\ell + a_2\ell^2 + \cdots + a_k\ell^k,$$

so that $N + 1 \equiv 1 \pmod{\ell}$. Now $N + 1 = \ell^{2a}(\ell b + r)$ for some $a, b \in \mathbb{N}_0$ implies that

$$N + 1 \equiv \begin{cases} 0 \pmod{\ell} & \text{if } a \geq 1, \\ r \pmod{\ell} & \text{if } a = 0, \end{cases}$$

so that

$$N + 1 \neq \ell^{2a}(\ell b + r) \text{ if } r \neq 1,$$
$$N + 1 = \ell^{2a}(\ell b + r) \text{ if } r = 1,$$

proving that

$$ct_{N+1}(F_2) - ct_N(F_2) = \begin{cases} 0 & \text{if } r \neq 1, \\ 1 & \text{if } r = 1. \end{cases} \tag{3.13}$$

From (1.2) we deduce

$$\mathcal{B}_{N+1}(j) = \begin{cases} \mathcal{B}_N(j) + 1 & \text{if } j = 1, \\ \mathcal{B}_N(j) & \text{if } j = 2, 3, \ldots, \ell - 1, \end{cases} \tag{3.14}$$

and

$$\mathcal{C}_{N+1}(j) = \mathcal{C}_N(j) \text{ if } j = 1, 2, 3, \ldots, \ell - 1. \tag{3.15}$$

By (3.11) we have

$$W(N+1) - W(N) = \frac{\ell}{\ell^2 - 1} + \sum_{j=r}^{\ell-1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) - \sum_{j=1}^{\ell-1} \left(\frac{j\ell}{\ell^2 - 1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) \right.$$
$$\left. - \frac{j}{\ell^2 - 1} (\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j)) \right).$$

Appealing to (3.14) and (3.15), we obtain

$$W(N+1) - W(N) = \begin{cases} 0 & \text{if } r \neq 1 \\ 1 & \text{if } r = 1. \end{cases} \tag{3.16}$$

Hence, by (3.13) and (3.16) we have

$$ct_{N+1}(F_2) - ct_N(F_2) = W(N+1) - W(N).$$

Case 2: $a_0 = 1, 2, \ldots, \ell - 2$.

In this case we have $\ell \geq 3$, $a_0 + 1 \neq \ell$ and

$$N = a_0 + a_1\ell + \cdots + a_k\ell^k, \ N + 1 = (a_0 + 1) + a_1\ell + \cdots + a_k\ell^k, \tag{3.17}$$

so that

$$N+1\begin{cases}\neq \ell^{2a}(\ell b+r) & \text{(for all } a,b\in\mathbb{N}_0\text{) if } r\neq a_0+1,\\ = \ell^{2a}(\ell b+r) & \text{(for } a=b=0\text{) if } r=a_0+1,\end{cases}$$

proving that

$$ct_{N+1}(F_2)-ct_N(F_2)=\begin{cases}0 & \text{if } r\neq a_0+1,\\ 1 & \text{if } r=a_0+1.\end{cases} \tag{3.18}$$

From (3.17) we deduce

$$\mathcal{B}_{N+1}(j)=\begin{cases}\mathcal{B}_N(j)-1 & \text{if } j=a_0,\\ \mathcal{B}_N(j)+1 & \text{if } j=a_0+1,\\ \mathcal{B}_N(j) & \text{if } j\neq a_0,a_0+1,\end{cases} \tag{3.19}$$

and

$$\mathcal{C}_{N+1}(j)=\mathcal{C}_N(j) \text{ if } j=1,2,\ldots,\ell-1. \tag{3.20}$$

Hence, by (3.11), (3.19) and (3.20), we obtain

$$W(N+1)-W(N)=\begin{cases}0 & \text{if } r\neq a_0+1\\ 1 & \text{if } r=a_0+1.\end{cases} \tag{3.21}$$

So, by (3.18) and (3.21) we have

$$ct_{N+1}(F_2)-ct_N(F_2)=W(N+1)-W(N).$$

Case 3: $a_0=a_1=\cdots=a_k=\ell-1$.

In this case, we have

$$N=(\ell-1)+(\ell-1)\ell+(\ell-1)\ell^2+\cdots+(\ell-1)\ell^k, \quad N+1=\ell^{k+1}.$$

We have

$$N+1=\ell^{2a}(\ell b+r) \text{ for some } a,b\in\mathbb{N}_0$$

if and only if

$$k \text{ is odd and } r=1.$$

Thus

$$ct_{N+1}(F_2)-ct_N(F_2)=\begin{cases}1 & \text{if } k \text{ odd, } r=1,\\ 0 & \text{if } k \text{ odd, } r\geq 2, \text{ or } k \text{ even.}\end{cases} \tag{3.22}$$

From (1.2) we see that for $j=0,1,2,\ldots,\ell-1$

$$\mathcal{B}_N(j)=\begin{cases}0 & \text{if } j\neq \ell-1,\\ k/2+1 & \text{if } j=\ell-1,\ k \text{ even},\\ (k+1)/2 & \text{if } j=\ell-1,\ k \text{ odd},\end{cases}$$

$$\mathcal{C}_N(j)=\begin{cases}0 & \text{if } j\neq \ell-1,\\ k/2 & \text{if } j=\ell-1,\ k \text{ even},\\ (k+1)/2 & \text{if } j=\ell-1,\ k \text{ odd},\end{cases}$$

and for $j = 1, 2, \cdots, \ell - 1$

$$\mathcal{B}_{N+1}(j) = \begin{cases} 1 & \text{if } j = 1,\ k \text{ odd}, \\ 0 & \text{if } j = 1,\ k \text{ even, or } j \neq 1, \end{cases}$$

and

$$\mathcal{C}_{N+1}(j) = \begin{cases} 1 & \text{if } j = 1,\ k \text{ even}, \\ 0 & \text{if } j = 1,\ k \text{ odd, or } j \neq 1, \end{cases}$$

The cases $\ell = 2$ and $\ell \neq 2$ work differently. In the case $\ell \neq 2$, as $\ell - 1 \neq 1$, for $j = 1, 2, \cdots \ell - 1$, we obtain

$$\sum_{j=r}^{\ell-1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} -(k-1)/2 & \text{if } r = 1,\ k \text{ odd}, \\ -(k+1)/2 & \text{if } r \geq 2,\ k \text{ odd}, \\ -k/2 - 1 & \text{if } k \text{ even}, \end{cases} \tag{3.23}$$

$$\sum_{j=1}^{\ell-1} \frac{j\ell}{\ell^2 - 1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} \dfrac{\ell}{\ell^2-1} - \dfrac{\ell^2-\ell}{\ell^2-1}\left(\dfrac{k+1}{2}\right) & \text{if } k \text{ odd}, \\ -\dfrac{\ell^2-\ell}{\ell^2-1}\left(\dfrac{k}{2}+1\right) & \text{if } k \text{ even}, \end{cases} \tag{3.24}$$

$$\sum_{j=1}^{\ell-1} \frac{j}{\ell^2 - 1} (\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j)) = \begin{cases} -\dfrac{\ell-1}{\ell^2-1}\left(\dfrac{k+1}{2}\right) & \text{if } k \text{ odd}, \\ \dfrac{1}{\ell^2-1} - \dfrac{\ell-1}{\ell^2-1}\left(\dfrac{k}{2}\right) & \text{if } k \text{ even}, \end{cases} \tag{3.25}$$

Thus appealing to (3.11), (3.23), (3.24) and (3.25), we deduce

$$\begin{aligned} W(N+1) - W(N) &= \frac{\ell}{\ell^2-1} + \begin{cases} -(k-1)/2 & \text{if } r = 1,\ k \text{ odd}, \\ -(k+1)/2 & \text{if } r \geq 2,\ k \text{ odd}, \\ -k/2 - 1 & \text{if } k \text{ even}, \end{cases} \\ &\quad - \begin{cases} \dfrac{\ell}{\ell^2-1} - \dfrac{\ell^2-\ell}{\ell^2-1}\left(\dfrac{k+1}{2}\right) - \dfrac{\ell-1}{\ell^2-1}\left(\dfrac{k+1}{2}\right) & \text{if } k \text{ odd}, \\ -\dfrac{\ell^2-\ell}{\ell^2-1}\left(\dfrac{k}{2}+1\right) + \dfrac{1}{\ell^2-1} - \dfrac{\ell-1}{\ell^2-1}\left(\dfrac{k}{2}\right) & \text{if } k \text{ even}, \end{cases} \\ &= \begin{cases} 1 & \text{if } k \text{ odd},\ r = 1, \\ 0 & \text{if } k \text{ odd},\ r \geq 2 \text{ or } k \text{ even}. \end{cases} \end{aligned} \tag{3.26}$$

We now turn to the case $\ell = 2$ in which case $\ell - 1 = 1$. We have

$$\mathcal{B}_N(j) = \begin{cases} (k+1)/2 & \text{if } j = 1,\ k \text{ odd}, \\ k/2 + 1 & \text{if } j = 1,\ k \text{ even}, \end{cases}$$

$$\mathcal{C}_N(j) = \begin{cases} (k+1)/2 & \text{if } j = 1,\ k \text{ odd}, \\ k/2 & \text{if } j = 1,\ k \text{ even}, \end{cases}$$

$$\mathcal{B}_{N+1}(j) = \begin{cases} 1 & \text{if } j = 1,\ k \text{ odd}, \\ 0 & \text{if } j = 1,\ k \text{ even}, \end{cases}$$

and

$$\mathcal{C}_{N+1}(j) = \begin{cases} 0 & \text{if } j = 1,\ k \text{ odd}, \\ 1 & \text{if } j = 1,\ k \text{ even}. \end{cases}$$

Thus for $\ell = 2$, as $r = 1$, we have

$$\sum_{j=r}^{\ell-1}(\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} -(k-1)/2 & \text{if } k \text{ odd,} \\ -k/2 - 1 & \text{if } k \text{ even,} \end{cases} \tag{3.27}$$

$$\sum_{j=1}^{\ell-1} \frac{j\ell}{\ell^2 - 1}(\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} \frac{2}{3}(-(k-1)/2) & \text{if } k \text{ odd,} \\ \frac{2}{3}(-k/2 - 1) & \text{if } k \text{ even,} \end{cases} \tag{3.28}$$

$$\sum_{j=1}^{\ell-1} \frac{j}{\ell^2 - 1}(\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j)) = \begin{cases} \frac{1}{3}(-(k+1)/2) & \text{if } k \text{ odd,} \\ \frac{1}{3}(-k/2 + 1) & \text{if } k \text{ even.} \end{cases} \tag{3.29}$$

Thus by (3.11), (3.27), (3.28) and (3.29) we deduce

$$\begin{aligned} W(N+1) - W(N) &= \begin{cases} \frac{2}{3} - \left(\frac{k-1}{2}\right) + \frac{2}{3}\left(\frac{k-1}{2}\right) + \frac{1}{3}\left(\frac{k+1}{2}\right) & \text{if } k \text{ odd,} \\ \frac{2}{3} - \left(\frac{k}{2} + 1\right) + \frac{2}{3}\left(\frac{k}{2} + 1\right) + \frac{1}{3}\left(\frac{k}{2} - 1\right) & \text{if } k \text{ even,} \end{cases} \\ &= \begin{cases} 1 & \text{if } k \text{ odd,} \\ 0 & \text{if } k \text{ even.} \end{cases} \end{aligned} \tag{3.30}$$

Hence, by (3.22), (3.26) and (3.30) we have

$$ct_{N+1}(F_2) - ct_N(F_2) = W(N+1) - W(N).$$

Case 4: If $a_0 = a_1 = \cdots = a_k = \ell - 1$, $a_{s+1} \neq \ell - 1$ for some s with $0 \leq s \leq k-1$, we have

$$\begin{aligned} N &= (\ell - 1) + (\ell - 1)\ell + \cdots + (\ell - 1)\ell^s + a_{s+1}\ell^{s+1} + \cdots + a_k\ell^k, \\ N + 1 &= (a_{s+1} + 1)\ell^{s+1} + \cdots + a_k\ell^k, \end{aligned}$$

where $a_{s+1} + 1 \neq \ell$ and $a_k \neq 0$.

We have

$$N + 1 = \ell^{2a}(\ell b + r) \text{ for some } a, b \in \mathbb{N}_0$$

if and only if

$$(a_{s+1} + 1)\ell^{s+1} + \cdots + a_k\ell^k = \ell^{2a}(\ell b + r)$$

if and only if

$$s + 1 = 2a,\ b = 0,\ r = a_{s+1} + 1$$

if and only if

$$s \text{ is odd and } r = a_{s+1} + 1.$$

Thus

$$ct_{N+1}(F_2) - ct_N(F_2) = \begin{cases} 0 & \text{if } s \text{ odd, } r \neq a_{s+1} + 1, \text{ or } s \text{ even,} \\ 1 & \text{if } s \text{ odd, } r = a_{s+1} + 1. \end{cases} \tag{3.31}$$

We let

$$M := a_{s+2}\ell^{s+2} + \cdots + a_k\ell^k,$$

$$N = (\ell-1) + (\ell-1)\ell + \cdots + (\ell-1)\ell^s + a_{s+1}\ell^{s+1} + M,$$
$$N+1 = (a_{s+1}+1)\ell^{s+1} + M,$$

where $a_{s+1} \neq \ell - 1$. Hence we have for $j = 0, 1, 2, \ldots, \ell - 1$

$$\mathcal{B}_N(j) = \begin{cases} \mathcal{B}_M(j) & \text{if } j = a_{s+1},\ s \text{ even}, \\ \mathcal{B}_M(j) + 1 & \text{if } j = a_{s+1},\ s \text{ odd}, \\ \mathcal{B}_M(j) + s/2 + 1 & \text{if } j = \ell - 1,\ s \text{ even}, \\ \mathcal{B}_M(j) + (s+1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ \mathcal{B}_M(j) & \text{if } j \neq a_{s+1},\ j \neq \ell - 1, \end{cases}$$

$$\mathcal{C}_N(j) = \begin{cases} \mathcal{C}_M(j) + 1 & \text{if } j = a_{s+1},\ s \text{ even}, \\ \mathcal{C}_M(j) & \text{if } j = a_{s+1},\ s \text{ odd}, \\ \mathcal{C}_M(j) + s/2 & \text{if } j = \ell - 1,\ s \text{ even}, \\ \mathcal{C}_M(j) + (s+1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ \mathcal{C}_M(j) & \text{if } j \neq a_{s+1},\ j \neq \ell - 1, \end{cases}$$

for $j = 1, 2, \ldots, \ell - 1$

$$\mathcal{B}_{N+1}(j) = \begin{cases} \mathcal{B}_M(j) + 1 & \text{if } j = a_{s+1} + 1,\ s \text{ odd}, \\ \mathcal{B}_M(j) & \text{if } j = a_{s+1} + 1,\ s \text{ even, or } j \neq a_{s+1} + 1, \end{cases}$$

$$\mathcal{C}_{N+1}(j) = \begin{cases} \mathcal{C}_M(j) + 1 & \text{if } j = a_{s+1} + 1,\ s \text{ even}, \\ \mathcal{C}_M(j) & \text{if } j = a_{s+1} + 1,\ s \text{ odd, or } j \neq a_{s+1} + 1. \end{cases}$$

Thus for $a_{s+1} \neq \ell - 2$ (so that $a_{s+1} + 1 \neq \ell - 1$) and $j = 1, 2, \ldots, \ell - 1$

$$\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j) = \begin{cases} -1 & \text{if } j = a_{s+1},\ s \text{ odd}, \\ 1 & \text{if } j = a_{s+1} + 1,\ s \text{ odd}, \\ -s/2 - 1 & \text{if } j = \ell - 1,\ s \text{ even}, \\ -(s+1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ 0 & \text{if } j = a_{s+1},\ s \text{ even, or } j = a_{s+1} + 1,\ s \text{ even}, \\ & \text{or } j \neq a_{s+1}, a_{s+1} + 1, \ell - 1, \end{cases}$$

$$\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j) = \begin{cases} -1 & \text{if } j = a_{s+1},\ s \text{ even}, \\ 1 & \text{if } j = a_{s+1} + 1,\ s \text{ even}, \\ -s/2 & \text{if } j = \ell - 1,\ s \text{ even}, \\ -(s+1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ 0 & \text{if } j = a_{s+1},\ s \text{ odd, or } j = a_{s+1} + 1,\ s \text{ odd}, \\ & \text{or } j \neq a_{s+1}, a_{s+1} + 1, \ell - 1, \end{cases}$$

whereas if $a_{s+1} = \ell - 2$ (so that $a_{s+1} + 1 = \ell - 1$) and $j = 1, 2, \ldots, \ell - 1$ we have

$$\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j) = \begin{cases} -1 & \text{if } j = \ell - 2,\ s \text{ odd}, \\ -s/2 - 1 & \text{if } j = \ell - 1,\ s \text{ even}, \\ -(s-1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ 0 & \text{if } j = \ell - 2,\ s \text{ even, or } j \neq \ell - 1, \ell - 2, \end{cases}$$

$$\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j) = \begin{cases} -1 & \text{if } j = \ell - 2,\ s \text{ even}, \\ -s/2 + 1 & \text{if } j = \ell - 1,\ s \text{ even}, \\ -(s+1)/2 & \text{if } j = \ell - 1,\ s \text{ odd}, \\ 0 & \text{if } j = \ell - 2,\ s \text{ odd, or } j \neq \ell - 1, \ell - 2. \end{cases}$$

Hence we obtain for $a_{s+1} \neq \ell - 2$

$$X := \sum_{j=r}^{\ell-1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} -s/2 - 1 & \text{if } s \text{ even}, \\ -s/2 - 1/2 & \text{if } s \text{ odd},\ r \neq a_{s+1} + 1, \\ -s/2 + 1/2 & \text{if } s \text{ odd},\ r = a_{s+1} + 1. \end{cases}$$

Also

$$Y := \sum_{j=1}^{\ell-1} \frac{j\ell}{\ell^2 - 1} (\mathcal{B}_{N+1}(j) - \mathcal{B}_N(j)) = \begin{cases} -\dfrac{\ell^2 - \ell}{\ell^2 - 1} \left(\dfrac{s}{2} + 1 \right) & \text{if } s \text{ even}, \\ \dfrac{\ell}{\ell^2 - 1} - \dfrac{\ell^2 - \ell}{\ell^2 - 1} \left(\dfrac{s+1}{2} \right) & \text{if } s \text{ odd}, \end{cases}$$

and

$$Z := \sum_{j=1}^{\ell-1} \frac{j}{\ell^2 - 1} (\mathcal{C}_{N+1}(j) - \mathcal{C}_N(j)) = \begin{cases} \dfrac{1}{\ell^2 - 1} - \dfrac{\ell - 1}{\ell^2 - 1} \left(\dfrac{s}{2} \right) & \text{if } s \text{ even}, \\ -\dfrac{\ell - 1}{\ell^2 - 1} \left(\dfrac{s+1}{2} \right) & \text{if } s \text{ odd}. \end{cases}$$

Thus

$$W(N+1) - W(N) = \frac{\ell}{\ell^2 - 1} + X - Y - Z = \begin{cases} 0 & \text{if } s \text{ even, or } s \text{ odd},\ r \neq a_{s+1} + 1, \\ 1 & \text{if } s \text{ odd},\ r = a_{s+1} + 1. \end{cases} \tag{3.32}$$

Hence, by (3.31) and (3.32) we have

$$ct_{N+1}(F_2) - ct_N(F_2) = W(N+1) - W(N).$$

Therefore, in each case, we have

$$ct_{N+1}(F_2) - ct_N(F_2) = W(N+1) - W(N).$$

By the inductive hypothesis (3.12) this means $ct_{N+1}(F_2) = W(N+1)$. This completes the proof of the formula (1.10).

Next, we prove the formula (1.11) for the form F_3. As $ct_N(F_2) + ct_N(F_3) = ct_N(F_1)$, we have

$$ct_N(F_3) = ct_N(F_1) - ct_N(F_2). \tag{3.33}$$

Then the formula (1.11) follows by using (1.9) and (1.10) in (3.33) and simplifying using (1.3).

This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. For $i \in \{1,3\}$ we have

$$ct_N(2^{2a}(8b+i)) + ct_N(2^{2a}(8b+4+i)) = ct_N(4^a(4b+i)),$$

whose formula is a special case of Theorem 1.1. Next, we give the formula for

$$ct_N(2^{2a}(8b+i)) - ct_N(2^{2a}(8b+4+i)).$$

Lemma 4.1. *We have*

$$ct_N(2^{2a}(8b+1)) - ct_N(2^{2a}(8b+5)) = \mathcal{E}_N(100) + \mathcal{E}_N(010) + \mathcal{E}_N(110) + \mathcal{E}_N(001), \tag{4.1}$$

$$ct_N(2^{2a}(8b+3)) - ct_N(2^{2a}(8b+7)) = \mathcal{E}_N(110) + \mathcal{E}_N(001) + \mathcal{E}_N(101) + \mathcal{E}_N(011). \tag{4.2}$$

Proof. If $n = 2^{2a}(8b+1)$ for some a and b, we call $n^* = 2^{2a}(8b+5)$, its companion. Let the expansion of N be given by

$$N = \sum_{j=0}^{k} a_j 2^j = (a_0 \dots a_k 00).$$

In the base 2 expansion of N, if one of the following patterns occurs

$$(a_{2i}a_{2i+1}a_{2i+2}) = (100), (010), (110) \text{ or } (001), \tag{4.3}$$

then there exists some $n \leq N$ with the base 2 expansion

$$n = 1 \cdot 2^{2i} + 0 \cdot 2^{2i+1} + 0 \cdot 2^{2i+2} + \sum_{j=2i+3}^{k} a_j 2^j.$$

This means n is of the form $2^{2i}(8b+1)$. However, its companion

$$n^* = 2^{2i}(8b+5) = 1 \cdot 2^{2i} + 0 \cdot 2^{2i+1} + 1 \cdot 2^{2i+2} + \sum_{j=2i+3}^{k} a_j 2^j$$

is greater than N. For all other $n \leq N$, of the form $2^{2a}(8b+1)$, its companion $n^* = 2^{2a}(8b+5)$ is less than or equal to N, that is, in these cases there is a one to one correspondence between $n \leq N$ and $n^* \leq N$. Therefore, the difference between $ct_N(2^{2a}(8b+1))$ and $ct_N(2^{2a}(8b+5))$ can be counted by the occurrence of the patterns given in (4.3). This proves (4.1).

The proof of (4.2) is similar by observing that the differences can be counted by the occurrence of one of the patterns

$$(a_{2i}a_{2i+1}a_{2i+2}) = (110), (001), (101) \text{ or } (011).$$

□

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. First we prove the formulas concerning $ct_N(2^{2a}(8b+r))$. Occurrence of 1 in the base 4 expansion of a number means that in base two expansion there is a pattern $(a_{2r}a_{2r+1}) = (10)$. This means we have

$$\mathcal{A}_{N,4}(1) = \mathcal{E}_N(100) + \mathcal{E}_N(101). \tag{4.4}$$

Similarly, we have

$$\mathcal{A}_{N,4}(2) = \mathcal{E}_N(010) + \mathcal{E}_N(011). \tag{4.5}$$

By Theorem 1.1, with $\ell = 4$, we have

$$ct_N(4^a(4b+1)) = ct_N(2^{2a}(8b+1)) + ct_N(2^{2a}(8b+5)) = \frac{N}{3} + \frac{2}{3}\mathcal{A}_{N,4}(1) + \frac{1}{3}\mathcal{A}_{N,4}(2),$$

which by (4.4) and (4.5) yields

$$ct_N(2^{2a}(8b+1)) + ct_N(2^{2a}(8b+5)) = \frac{N}{3} + \frac{2}{3}(\mathcal{E}_N(100) + \mathcal{E}_N(101)) + \frac{1}{3}(\mathcal{E}_N(010) + \mathcal{E}_N(011)).$$

By Lemma 4.1 we have

$$ct_N(2^{2a}(8b+1)) - ct_N(2^{2a}(8b+5)) = \mathcal{E}_N(100) + \mathcal{E}_N(010) + \mathcal{E}_N(110) + \mathcal{E}_N(001).$$

Adding the two equations above side by side and dividing the resulting equation by 2, we obtain the desired formula for $ct_N(2^{2a}(8b+1))$.

The formula for $ct_N(2^{2a}(8b+5))$ follows similarly by subtracting the above equations.

The proofs of the formulas for $ct_N(2^{2a}(8b+3))$ and $ct_N(2^{2a}(8b+7))$ are similar.

Next we prove the formulas concerning $ct_N(2^{2a+1}(8b+r))$. Let n be of the form $2^{2a+1}(8b+1)$ and let $n^* = n/2$. Clearly, n^* is of the form $2^{2a}(8b+1)$. We have $n \le N$ if and only if $n^* \le (N-a_0)/2$. Therefore, to count the number of $n \le N$ of the form $2^{2a+1}(8b+1)$ we count the number of $n^* \le (N-a_0)/2$ of the form $2^{2a}(8b+1)$. Hence, by using the previously proven formula for $ct_N(2^{2a}(8b+1))$ we have

$$\begin{aligned} ct_N(2^{2a+1}(8b+1)) &= ct_{(N-a_0)/2}(2^{2a}(8b+1)) \\ &= \frac{(N-a_0)/2}{6} + \frac{5}{6}\mathcal{E}_{(N-a_0)/2}(100) + \frac{2}{3}\mathcal{E}_{(N-a_0)/2}(010) + \frac{1}{2}\mathcal{E}_{(N-a_0)/2}(110) \\ &\quad + \frac{1}{2}\mathcal{E}_{(N-a_0)/2}(001) + \frac{1}{3}\mathcal{E}_{(N-a_0)/2}(101) + \frac{1}{6}\mathcal{E}_{(N-a_0)/2}(011). \end{aligned}$$

In base 2 expansion, the digits of N are digits of $(N-a_0)/2$ shifted by 1 place. Therefore, even indices of $(N-a_0)/2$ are odd indices of N. Hence, we have

$$\begin{aligned} ct_N(2^{2a+1}(8b+1)) = \frac{(N-a_0)/2}{6} &+ \frac{5}{6}\mathcal{O}_N(100) + \frac{2}{3}\mathcal{O}_N(010) + \frac{1}{2}\mathcal{O}_N(110) \\ &+ \frac{1}{2}\mathcal{O}_N(001) + \frac{1}{3}\mathcal{O}_N(101) + \frac{1}{6}\mathcal{O}_N(011). \end{aligned}$$

Finally, the asserted formula for $ct_N(2^{2a+1}(8b+1))$ follows by observing that

$$a_0 = \begin{cases} 1 & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

The proofs of formulas for $ct_N(2^{2a+1}(8b+3))$, $ct_N(2^{2a+1}(8b+5))$, and $ct_N(2^{2a+1}(8b+7))$ are similar. □

References

[1] M. C. Chakrabarti, *On the limit points of a function connected with the three-square problem*, Bull. Calcutta Math. Soc. **32** (1940), 1–6.

[2] G. Doyle, J. B. Muskat, L. Pehlivan and K. S. Williams, *Positive integers represented by regular primitive positive-definite integral ternary quadratic forms*, Integers **19** (2019), Paper No. A45.

[3] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer-Verlag, New York, 1985. Doi: 10.1007/978-1-4613-8566-0.

[4] E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. Math. Phys. (3) **13** (1908), 305–312.

[5] P. Shiu, *Counting sums of three squares*, Bull. London Math. Soc. **20** (1988), 203–208. Doi: 10.1112/blms/20.3.203.

Zafer Selcuk Aygin
Department of Mathematics and Statistics,
American University of Sharjah,
Sharjah,
UAE
saygin@aus.edu

Kenneth S. Williams
School of Mathematics and Statistics,
Carleton University,
Ottawa, Ontario K1S 5B6,
Canada
kennethwilliams@cunet.carleton.ca